Course 1

Monday, October 4, 2021 6:30 PM

Final grade = 40% seminar activities + 60% Final exam (courses 1-13)

$\underbrace{}$ 10% answers during pr.-sol. + 30% Test during the seminar (courses 1-6)

↓ 18th Nov.

Bibliography

1. G. Kohr „Complex Analysis", lecture notes, 2020
2. D. Hamburg, P. Mocanu, N. Negoescu „Mathematical Analysis (Complex Functions)", 1982
3. G. Kohr, P. Mocanu : „Special Topics of C.A.", 2005
4. J. Conway : „Functions of one complex variable, I", 1978
5. W. Rudin : „Real and Complex Analysis", 1987

1) M. Beck et al. : „A First Course in C.A.", http://math.sfsu.edu/beck/complex.html
2) M. Bonk : „C.A.", course notes, https://math.ucla.edu/~mbonk/

https://www.math.ucla.edu/~mbonk/246c.1.12s/complana

Complex numbers

Consider the following operations on $\mathbb{R}^2$:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1), \quad (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.$$

$(\mathbb{R}^2, +, \cdot)$ becomes a commutative field, which we denote by $\mathbb{C}$ and we call it the field of complex numbers.

$(\mathbb{R} \times \{0\}, +, \cdot)$ is a subfield of $\mathbb{C}$ $((x,0)\cdot(y,0) = (xy, 0), x, y \in \mathbb{R})$ and $\varphi : \mathbb{R} \to \mathbb{R} \times \{0\}$ given by $\varphi(x) = (x, 0)$, $x \in \mathbb{R}$, is isomorphism, where we take the usual addition and multiplication on $\mathbb{R}$.

We identify $\mathbb{R} \times \{0\}$ with $\mathbb{R}$, $(x, 0) \equiv x$, $x \in \mathbb{R}$. So, $\mathbb{R} \subset \mathbb{C}$.

Denote $i = (0, 1)$ (the imaginary unit). Then, $\forall (x, y) \in \mathbb{C}$:

$$(x, y) = (x, 0) + (0, y) = x + (y, 0) \cdot (0, 1)$$

$(x,y) = x + y\cdot i$.

So, $\forall z\in\mathbb{C}: \exists!\, x,y\in\mathbb{R}$ s.t. $\boxed{z = x+i\cdot y}$.

(!) $i^2 = (0,1)\cdot(0,1) = (-1,0) = -1 \Rightarrow i^2 = -1$.

(!) $\mathbb{C}$ is not an ordered field!

The algebraic form

Let $z = x+iy \in\mathbb{C}$. Denote: $\operatorname{Re} z = x$ (real part), $\operatorname{Im} z = y$ (imaginary part),

$|z| = \sqrt{x^2+y^2}$ (modulus), $\bar z = x - iy$ (conjugate).

(!) $z = \operatorname{Re} z + i\operatorname{Im} z$; $\operatorname{Re} z = \dfrac{z+\bar z}{2}$, $\operatorname{Im} z = \dfrac{z-\bar z}{2i}$;

$|z| = |\bar z|$; $|z|^2 = z\cdot\bar z$; $\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} \le |z| \le |\operatorname{Re} z| + |\operatorname{Im} z|$;

$|z_1+z_2| \le |z_1| + |z_2|$ (triangle ineq.); $|z_1 z_2| = |z_1|\,|z_2|$;

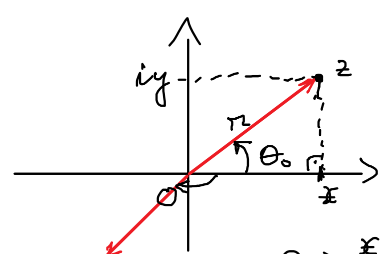
$z\neq 0$, $\dfrac{1}{z} = \dfrac{\bar z}{|z|^2} = \dfrac{x}{x^2+y^2} + i\dfrac{-y}{x^2+y^2}$ (so, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is indeed a group).

The trigonometric form

Let $z\in\mathbb{C}^*$, $r = |z| > 0$.

Every $\theta\in\mathbb{R}$ that satisfies $(*)$ $\cos\theta + i\sin\theta = \dfrac{z}{|z|}$

is called an argument of z.



$\cos\theta_0 = \frac{x}{r}$

$\sin\theta_0 = \frac{y}{r}$

$(r,\theta_0) \to$ polar coordinates

(!) $z=0$ has no argument.

$(*) \Longleftrightarrow \underbrace{\operatorname{Re} z}_{x} = r\cos\theta$, $\underbrace{\operatorname{Im} z}_{y} = r\sin\theta$.

$\exists!\,\theta_0\in(-\pi,\pi]$ satisfies $(*)$ which is called the principal value of the argument of z.

Notation: $\arg z = \theta_0 \in (-\pi,\pi]$.

$\arg z$ is the angle between $\overrightarrow{Oz}$ and positive real semiaxis.

Note that $\forall$ argument θ of z $\exists k \in \mathbb{Z}$ s.t.

$$\theta = \theta_0 + 2k\pi \quad \left(\text{i.e., } \theta = \theta_0 \pmod{2\pi} \right).$$

Let $\operatorname{Arg} z = \{ \arg z + 2k\pi : k \in \mathbb{Z} \}$ be the set of all the arguments of z.

$\operatorname{Arg} : \mathbb{C}^* \to \mathcal{P}(\mathbb{R})$ is the multi-valued argument function.

(!) $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$, $z_1, z_2 \in \mathbb{C}^*$.

$(\Longleftrightarrow \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi})$

$\operatorname{Arg} \dfrac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$

$\operatorname{Arg} \dfrac{1}{z} = -\operatorname{Arg} z$, $\operatorname{Arg} \bar{z} = -\operatorname{Arg} z$, $z \in \mathbb{C}^*$

$(\operatorname{Arg} 1 = \{2k\pi ; k \in \mathbb{Z}\}$, $\operatorname{Arg}(rz) = \operatorname{Arg} z$, $\forall r > 0)$

Ex: $z_1 = z_2 = -1 \Rightarrow \arg z_1 = \arg z_2 = \pi$

$\arg(z_1 z_2) = \arg 1 = 0$

$\arg(z_1 z_2) \neq \arg z_1 + \arg z_2$.

(!) $\forall z \in \mathbb{C}^*$: $z = r(\cos\theta + i \sin\theta)$, $r = |z|$, $\theta \in \operatorname{Arg} z$

$\hookrightarrow$ is called the trigonometric form of z.

(!) $z > 0 \Longleftrightarrow \arg z = 0$. $z \in \mathbb{R}^* \Longleftrightarrow \arg z \in \{0, \pi\}$.

$z < 0 \Longleftrightarrow \arg z = \pi$.

z is on the imaginary axis $\Longleftrightarrow \arg z \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$.

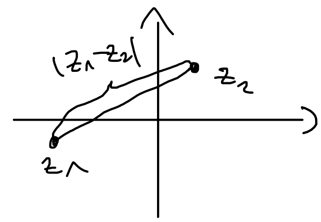
The topology of the complex plane

Each point $P(x, y) \in \mathbb{R}^2$ is identified with the complex number $z = x + iy$, called the affix of P.

The complex plane is the set of complex numbers equipped with Euclidean structure of $\mathbb{R}^2$. We denote it again by $\mathbb{C}$.

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in \mathbb{C}$, then the Euclidean distance between z_1 and z_2 is

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$



Denote, for $z_0 \in \mathbb{C}$, $r > 0$:

$U(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$ open disk;

$\overline{U}(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| \leq r \}$ closed disk;

$\partial U(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| = r \}$ circle;

$\dot{U}(z_0, r) = \{ z \in \mathbb{C} : 0 < |z - z_0| < r \} = U(z_0, r) \setminus \{z_0\}$ punctured disk;

$U(z_0; r_1, r_2) = \{ z \in \mathbb{C} : r_1 < |z - z_0| < r_2 \}$ annulus.

$0 \leq r_1 < r_2$

(!) $U(z_0; 0, r) = \dot{U}(z_0, r)$.

Let $A \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$.

Recall from Calculus:

z_0 is:
- an interior point of A, if $\exists r > 0$ s.t. $U(z_0, r) \subseteq A$;
- a closure point of A, if $\forall r > 0 : U(z_0, r) \cap A \neq \emptyset$;
- a limit point of A, if $\forall r > 0 : \dot{U}(z_0, r) \cap A \neq \emptyset$;
- a boundary point of A, if $\forall r > 0$: $U(z_0, r) \cap A \neq \emptyset$, $U(z_0, r) \cap (\mathbb{C} \setminus A) \neq \emptyset$.

A is:
- open, if $\forall z \in A$ is an interior point of A
- closed, if $\forall z \in A$ is a closure point of A
- bounded, if $\exists r > 0$ s.t. $A \subseteq U(0, r)$.

- int A is the interior of A
- $\overline{A}$ is the closure of A
- ∂A is the boundary of A
- A' is the set of accumulation points of A.