Course10

Monday, December 13, 2021 6:30 PM

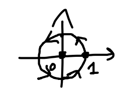
Complex integrals and primitives

$\boxed{T1}$ (connection between primitives and integrals on contours)

If $D \subseteq \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is continuous, then:
f has primitives on $D \iff \int_\gamma f = 0$, $\forall \gamma$ contour in D.

$\boxed{Ex.1}$ $f : \mathbb{C}^* \to \mathbb{C}$, $f(z) = \frac{1}{z}$, $z \in \mathbb{C}^*$.
$f \in \mathcal{H}(\mathbb{C}^*)$, but f has no primitives on $\mathbb{C}^*$.

Proof: Let $\gamma : [0,1] \to \mathbb{C}^*$, $\gamma(t) = e^{2\pi i t}$, $t \in [0,1]$.



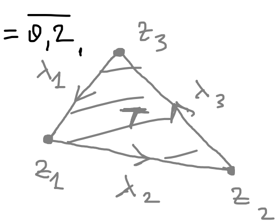
$$\int_\gamma f = \int_\gamma \frac{1}{z}\, dz \underset{\substack{\uparrow \\ \text{Ex.3,} \\ \text{Course3}}}{=} 2\pi i \neq 0 \overset{\boxed{T1}}{\Longrightarrow} f \text{ has no primitives.}$$

$\boxed{Ex.2}$ $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, $f(z) = \frac{1}{z}$, $z \in \mathbb{C} \setminus (-\infty, 0]$.
f has a primitive on $\mathbb{C}\setminus(-\infty,0]$: $\log : \mathbb{C}\setminus(-\infty,0] \to \mathbb{C}$
(the principal branch of Log on $\mathbb{C}\setminus(-\infty,0]$),

$$\log z = \ln|z| + i \arg z, \quad z \in \mathbb{C}\setminus(-\infty,0],$$
$$\log' z = \frac{1}{z}, \quad z \in \mathbb{C}\setminus(-\infty,0] \text{ (see Course 7).}$$

$\boxed{T2}$ (Cauchy's theorem for triangles)

Let $z_1, z_2, z_3 \in \mathbb{C}$ and $\Delta = (t_0 < t_1 < t_2 < t_3) \in \mathrm{Div}[a,b]$.
Let $\lambda_{k+1}(t) = \frac{t_{k+1} - t}{t_{k+1} - t_k} \cdot z_k + \frac{t - t_k}{t_{k+1} - t_k} \cdot z_{k+1}$, $t \in [t_k, t_{k+1}]$, $k = \overline{0,2}$.
$(z_0 = z_3)$

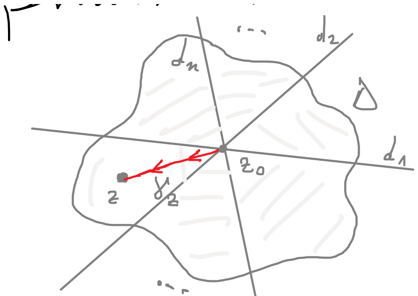


If $\Delta = \lambda_1 \cup \lambda_2 \cup \lambda_3$, T is the convex hull of z_1, z_2, z_3
and $f : T \to \mathbb{C}$ is continuous on T and
holomorphic on $\operatorname{int} T$, then $\int_\Delta f = 0$.

$\boxed{T3}$ Let $D \subseteq \mathbb{C}$ be a domain starlike w.r.t. $z_0 \in D$ and
$d_1, \dots, d_n$ be lines in $\mathbb{C}$ passing through z_0.
If $d = \bigcup_{k=1}^{n} d_k$ and $f : D \to \mathbb{C}$ continuous on D and
holomorphic on $D \setminus d$, then f has primitives on D.

Proof: Let for $z \in D$, $\gamma(t) = (1-t) z_0 + t z$, $t \in [0,1]$.

holomorphic on $D \setminus d$, then f has a primitive ...



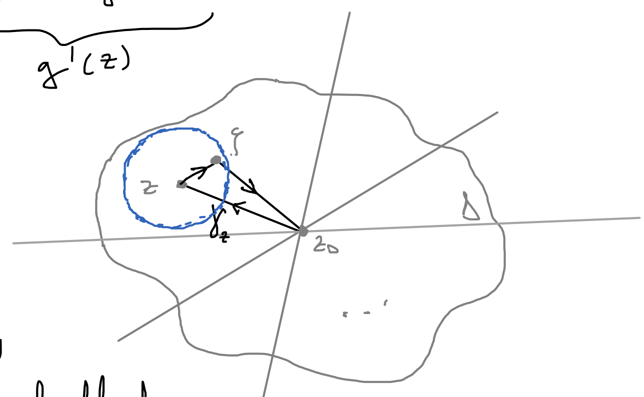
Proof: Let, for $z \in D$, $\gamma_z(t) = (1-t)z_0 + tz$, $t \in [0,1]$.

$\gamma_z \in C^1([0,1], D)$ is well-defined because D is starlike w.r.t. z_0.

Let $g : D \to \mathbb{C}$, $\underline{g(z) = \int_{\gamma_z} f}$, $z \in D$.

We want to prove that g is a primitive of f on D:

$$\forall z \in D : \underbrace{\lim_{\zeta \to z} \frac{g(\zeta) - g(z)}{\zeta - z}}_{g'(z)} = f(z).$$



Fix $z \in D$ arbitrary.

For $\zeta \in D$, let

$\lambda_{z\zeta}(t) = (2-t)z + (t-1)\zeta$, $t \in [1,2]$,

$\lambda_{\zeta z_0}(t) = (3-t)\zeta + (t-2)z_0$, $t \in [2,3]$,

$\Delta_\zeta = \underline{\gamma_z} \cup \underline{\lambda_{z\zeta}} \cup \underline{\lambda_{\zeta z_0}}$, T_ζ the conv. hull of z_0, z, ζ.

Case 1: $z \in D \setminus d \Rightarrow \exists r > 0$ s.t. $\forall \zeta \in U(z,r)$: int $T_\zeta \subset D \setminus d$,

$f \in \mathcal{H}(D \setminus d)$, $f \in C(D)$ $\xRightarrow{\boxed{T_2}}$ $0 = \int_{\Delta_\zeta} f = \int_{\gamma_z} f + \int_{\lambda_{z\zeta}} f + \int_{\lambda_{\zeta z_0}} f$

$= g(z) + \int_1^2 f(\lambda_{z\zeta}(t)) \cdot \lambda'_{z\zeta}(t)\,dt + \int_{\gamma_\zeta^-} f$

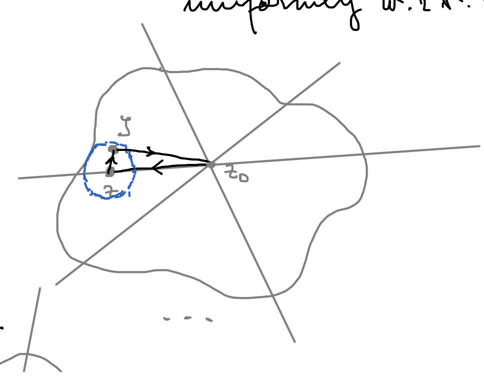
($\lambda_{\zeta z_0}$ is equivalent with γ_ζ^-)

$= g(z) - g(\zeta) + (\zeta - z) \int_1^2 f(\lambda_{z\zeta}(t))\,dt$

$\Rightarrow \dfrac{g(\zeta) - g(z)}{\zeta - z} = \int_1^2 f((2-t)z + (t-1)\zeta)\,dt \longrightarrow f(z)$, as $\zeta \to z$

($f((2-t)z + (t-1)\zeta) \longrightarrow f(z)$, as $\zeta \to z$) uniformly w.r.t. $t \in [1,2]$

$\Rightarrow \lim_{\zeta \to z} \dfrac{g(\zeta) - g(z)}{\zeta - z} = f(z)$.

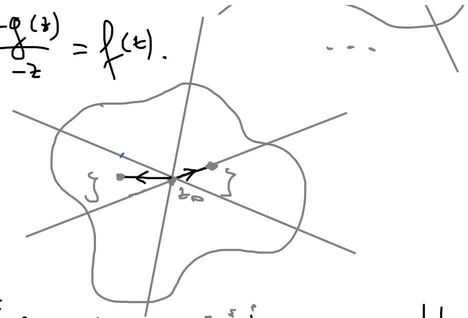


Case 2: $z \in (D \cap d) \setminus \{z_0\}$

$\Rightarrow \exists r > 0$ s.t. $\forall \zeta \in U(z,r)$: int $T_\zeta \subset D \setminus d$

$\xRightarrow{T_2}$ ... $\Rightarrow \lim_{\zeta \to z} \dfrac{g(\zeta) - g(z)}{\zeta - z} = f(z)$.

(ii) $\Rightarrow \dots \Rightarrow \lim\limits_{\zeta \to z} \dfrac{g(\zeta)-g(z)}{\zeta - z} = f(z).$



Case 3 : $z = z_0$

$\forall \zeta \in D \setminus \{z_0\}$:

$$\frac{g(\zeta)-g(z_0)}{\zeta - z_0} = \frac{1}{\zeta - z_0}\left(\int_0^1 f((1-t)z_0 + t\zeta)\cdot(\zeta - z_0)\,dt - \underbrace{\int_0^1 f(z_0)(z_0 - z_0)\,dt}_{=0} \right)$$

$$= \int_0^1 f((1-t)z_0 + t\zeta)\,dt \longrightarrow f(z_0), \text{ as } \zeta \to z_0$$

$$\Rightarrow \lim_{\zeta \to z_0} \frac{g(\zeta)-g(z_0)}{\zeta - z_0} = f(z_0).$$

$\boxed{T_4}$ Let $D \subseteq \mathbb{C}$ be a domain starlike w.r.t. $z_0 \in D$ and $\{z_1, \dots, z_m\} \subset D \setminus \{z_0\}$. If $f : D \to \mathbb{C}$ is continuous on D and holomorphic on $D \setminus \{z_0, z_1, \dots, z_m\}$, then f has primitives on D.

$\boxed{\text{Ex.3}}$ $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \begin{cases} \dfrac{\sin z}{z}, & z \in \mathbb{C}^* \\ 1, & z = 0. \end{cases}$

$f \in \mathcal{H}(\mathbb{C}^*)$. Since $\lim\limits_{z \to 0} f(z) = \lim\limits_{z\to 0} \dfrac{\sin z - \sin 0}{z - 0} = \sin' 0 = \cos 0 = 1 = f(0)$,

f is cont. on $\mathbb{C}$. $\boxed{T_4} \Rightarrow f$ has primitives on $\mathbb{C}$.

$\boxed{T_5}$ (fundamental theorem of Cauchy)

Let $D \subseteq \mathbb{C}$ be a domain starlike w.r.t. $z_0 \in D$ and $f \in \mathcal{H}(D)$. Then f has primitives on D.

In particular, $\int_\gamma f = 0$, $\forall \gamma$ contour in D.

$\boxed{R_1}$ Let $D \subseteq \mathbb{C}$ be a domain. D is <u>simply connected</u>, if

$\forall f : \partial U(0,1) \to D$ continuous, $\exists F : \overline{U}(0,1) \to D$ continuous, s.t. $F|_{\partial U(0,1)} = f$.

(!) D is simply connected $\Leftrightarrow \mathbb{C}_\infty \setminus D$ connected.

(!) D is starlike $\Rightarrow D$ is simply connected.

(!) $\boxed{T_5}$ holds for D simply connected.

Cauchy's integral formulae

$\boxed{P1}$ Let γ be a contour in $\mathbb{C}$, $\varphi : \{\gamma\} \to \mathbb{C}$ continuous and $\phi : \mathbb{C} \setminus \{\gamma\} \to \mathbb{C}$,

$$\phi(z) = \int_\gamma \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \mathbb{C} \setminus \{\gamma\}.$$

Then ϕ is infinitely differentiable on $\mathbb{C} \setminus \{\gamma\}$ and

$$\phi^{(n)}(z) = n! \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad z \in \mathbb{C} \setminus \{\gamma\}.$$

$\boxed{C1}$ Let $z_0 \in \mathbb{C}$, $r > 0$, $\gamma_r(t) = z_0 + r e^{2\pi i t}$, $t \in [0,1]$.

Then
$$\int_{\gamma_r} \frac{1}{\zeta - z}\, d\zeta = \begin{cases} 2\pi i, & z \in U(z_0, r) \\ 0, & z \in \mathbb{C} \setminus \overline{U}(z_0, r). \end{cases}$$

Proof: at seminar.

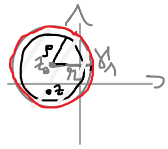
$\boxed{T6}$ (Cauchy's integral formulae)

Let $z_0 \in \mathbb{C}$, $r > 0$ and $f : \overline{U}(z_0, r) \to \mathbb{C}$ be continuous on $\overline{U}(z_0, r)$ and holomorphic on $U(z_0, r)$. Then f is infinitely differentiable on $U(z_0, r)$ and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad z \in U(z_0, r), \ n \in \mathbb{N},$$

where $\gamma_r(t) = z_0 + r e^{2\pi i t}$, $t \in [0,1]$.

Proof: Case $n = 0$: Let $z \in U(z_0, r)$ and $g_z : \overline{U}(z_0, r) \to \mathbb{C}$,



$$g_z(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \zeta \in \overline{U}(z_0, r) \setminus \{z\} \\ f'(z), & \zeta = z. \end{cases}$$

g_z is continuous on $\overline{U}(z_0, r)$ and holomorphic on $U(z_0, r) \setminus \{z\}$

$\overset{T_4}{\Longrightarrow}$ g_z has primitives on $U(z_0, r)$ $\overset{T_1}{\Longrightarrow}$ $\int_{\gamma_\rho} g_z = 0$, $\forall \rho \in (0, r)$,

where $\gamma_\rho(t) = z_0 + \rho e^{2\pi i t}$, $t \in [0,1]$,

$$\int_{\gamma_\rho} g_z = \int_0^1 g_z(z_0 + \rho e^{2\pi i t}) \cdot \rho \cdot (2\pi i) \cdot e^{2\pi i t}\, dt \xrightarrow[\rho \to r]{} \int_0^1 g_z(z_0 + r e^{2\pi i t}) \cdot r (2\pi i) e^{2\pi i t}\, dt$$

$\gamma_\rho(t) = z_0 + \rho e^{2\pi i t}$, $t \in [0,1]$,

$$\underbrace{\int_{\gamma_\rho} g_z}_{=0,\ \forall \rho \in (0,r)} = \int_0^1 g_z(z_0+\rho e^{2\pi i t})\cdot \rho\cdot(2\pi i)\cdot e^{2\pi i t}\,dt \xrightarrow[\rho\to r]{} \underbrace{\int_0^1 g_z(z_0+r e^{2\pi i t})\cdot r(2\pi i)\, e^{2\pi i t}\,dt}_{\int_{\gamma_r} g_z}$$

$$\Longrightarrow \int_{\gamma_r} g_z = 0 \Longrightarrow \int_{\gamma_r} \frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta = 0 \Longrightarrow \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)\underbrace{\int_{\gamma_r}\frac{1}{\zeta - z}\,d\zeta}_{2\pi i,\ \text{by } \boxed{C1}}.$$

<u>Case $n \in \mathbb{N}^*$</u>: Let $\phi : \mathbb{C}\setminus\{\gamma_r\} \to \mathbb{C}$,

$$\phi(z) = \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in \mathbb{C}\setminus\{\gamma_r\}.$$

f is cont. on $\{\gamma_r\} = \partial U(z_0,r) \overset{\boxed{P1}}{\Longrightarrow} \phi$ is infinitely diff. on $\mathbb{C}\setminus\{\gamma_r\}$

and $\phi^{(n)}(z) = n! \displaystyle\int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$, $z \in \mathbb{C}\setminus\{\gamma_r\}$.

Since $f(z) \overset{n=0}{=} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\phi(z)$, $z \in U(z_0,r)$,

f is infinitely differentiable on $U(z_0,r)$ and

$$f^{(n)}(z) = \frac{1}{2\pi i}\phi^{(n)}(z) = \frac{n!}{2\pi i}\int_{\gamma_r}\frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta, \quad z \in U(z_0,r).$$