Applications of the Cauchy formulae

$\boxed{C_1}$ Let $G \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(G)$.
Then f is infinitely differentiable on G.

Proof: use T_6, Course 10, locally on G.

$\boxed{C_2}$ (Morera)
Let $G \subseteq \mathbb{C}$ be open. If $f: G \to \mathbb{C}$ has primitives on G, then $f \in \mathcal{H}(G)$.

Proof: use C_1.

$\boxed{C_3}$ Let $D \subseteq \mathbb{C}$ be a domain starlike w.r.t. $z_0 \in D$ and $f: D \to \mathbb{C}$.
Then: f has primitives on $D \iff f \in \mathcal{H}(D)$.

Proof: use C_2 and $\boxed{T_5}$, Course 10.

$\boxed{C_4}$ Let $G \subseteq \mathbb{C}$ be open, $\{z_1, \dots, z_n\} \subset G$ and $f: G \to \mathbb{C}$ cont. on G and holomorphic on $G \setminus \{z_1, \dots, z_n\}$. Then $f \in \mathcal{H}(G)$.

Proof: use C_2 and $\boxed{T_4}$, Course 10.

$\boxed{C_5}$ (Cauchy's inequalities)
Let $z_0 \in \mathbb{C}$, $r > 0$, $f: \overline{U}(z_0, r) \to \mathbb{C}$ be cont. on $\overline{U}(z_0, r)$ and holomorphic $U(z_0, r)$. Then:

$$|f^{(n)}(z_0)| \le \frac{n!}{r^n} \cdot M, \quad \forall n \in \mathbb{N},$$

where $M = \max\limits_{\zeta \in \partial U(z_0, r)} |f(\zeta)|$.

Proof: use $\boxed{T_6}$, Course 10, and $\boxed{P_1}$, Course 9.

$\boxed{C_6}$ (Liouville) If f is entire $(f \in \mathcal{H}(\mathbb{C}))$ and bounded, then f is constant.

Proof: $C_5 \Rightarrow |f'(z)| \le \frac{1}{r} \cdot M$, $\forall z \in \mathbb{C}$, $\forall r > 0$, where $M = \sup\limits_{\zeta \in \mathbb{C}} |f(\zeta)| < \infty$

$\Rightarrow f' \equiv 0 \underset{\boxed{T_1},\ \text{Course 6}}{\Longrightarrow} f$ is constant on $\mathbb{C}$.

$\boxed{C_7}$ (fundamental theorem of algebra)
Let $p: \mathbb{C} \to \mathbb{C}$, $p(z) = a_0 + a_1 z + \dots + a_n z^n$, $z \in \mathbb{C}$,
where $a_k \in \mathbb{C}$, $k = \overline{0, n}$, $a_n \neq 0$, $n \in \mathbb{N}^*$.

Then $\exists z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$.

__Proof__: Assume that $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Let $f = \frac{1}{p} \in \mathcal{H}(\mathbb{C})$.

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{z^n\left(\underset{\underset{0}{\neq}}{a_n} + a_{n-1}\cdot \underset{\underset{0}{\downarrow}}{\frac{1}{z}} + \ldots + a_0 \cdot \underset{\underset{0}{\downarrow}}{\frac{1}{z^n}}\right)} = 0$$

$$\left.\begin{array}{l} \Rightarrow \exists \delta > 0 \text{ s.t. } |f(z)| < 1, \ \forall |z| > \delta \\ f \text{ is cont. on } \overline{U}(0,\delta) \Rightarrow f \text{ is bounded on } \overline{U}(0,\delta) \end{array}\right\} \Rightarrow f \text{ is bounded on } \mathbb{C} \overset{\boxed{L}}{\Rightarrow} f \text{ is constant on } \mathbb{C}$$

$\Rightarrow p$ is constant on $\mathbb{C}$
contradiction.

__Sequences of holomorphic functions and power series__

$\boxed{D1}$ Let $G \subseteq \mathbb{C}$ be open and $f_n : G \to \mathbb{C}$, $n \in \mathbb{N}$.

$(f_n)_{n\in\mathbb{N}}$ converges uniformly on compacts of G to $f : G \to \mathbb{C}$ $\left(f_n \xrightarrow[G]{u.c.} f \right)$, if

$\forall K \subset G : (f_n)_{n\in\mathbb{N}}$ converges uniformly on K to f

$(\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon, \forall z \in K, \forall n \geq N)$.

$\boxed{T1}$ (Weierstrass)

Let $G \subseteq \mathbb{C}$ be open and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}(G)$ s.t. $f_n \xrightarrow[G]{u.c.} f$ $(f : G \to \mathbb{C})$. Then $f \in \mathcal{H}(G)$ and

$$f_n^{(k)} \xrightarrow[G]{u.c.} f^{(k)}, \ \forall k \in \mathbb{N}.$$

$\boxed{T2}$ (Cauchy–Hadamard)

Consider the __power series__ around $z_0 \in \mathbb{C}$:

$$(*) \quad \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ where } a_n \in \mathbb{C},$$

w.r.t. the variable $z \in \mathbb{C}$,

and $R \in [0, \infty]$ given by

$$(**) \quad \frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}},$$ called the __radius of convergence__ of $(*)$.

If $R \in (0, \infty]$, then $(*)$ converges uniformly on compacts of $U(z_0, R)$ to a __holomorphic__ function $S : U(z_0, R) \to \mathbb{C}$,

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ z \in U(z_0, R).$$

Moreover, $S^{(k)}(z) = \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdot \ldots \cdot (n-k+1)(z-z_0)^{n-k}$, $z \in U(z_0,R)$, $k \in \mathbb{N}^*$.

$\boxed{R1}$ 1) If $R = \infty$, then $U(z_0,R)$ becomes $\mathbb{C}$ in T_2.

2) If $R \in (0,\infty]$, then $a_k = \frac{S^{(k)}(z_0)}{k!}$, $k \in \mathbb{N}$,

so $\sum_{n=0}^{\infty} a_n (z-z_0)^n = S(z) = \sum_{n=0}^{\infty} \frac{S^{(n)}(z_0)}{n!} \cdot (z-z_0)^n$, $\forall z \in U(z_0,R)$.

3) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l \Rightarrow R = \frac{1}{l}$.

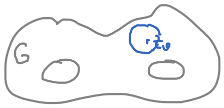
$\boxed{T3}$ (Taylor series expansion)

Let $z_0 \in \mathbb{C}$, $r > 0$ and $f \in \mathcal{H}(U(z_0,r))$. Then:

$\exists!$ power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with radius of convergence $R \geq r$

s.t. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $z \in U(z_0,r)$.

Moreover, $a_n = \frac{f^{(n)}(z_0)}{n!}$, $n \in \mathbb{N}$.



$\boxed{R2}$ Let $G \subseteq \mathbb{C}$ be open and $f: G \to \mathbb{C}$.

By T_2 and T_3, we have that: $f \in \mathcal{H}(G) \Longleftrightarrow$

$\Longleftrightarrow f$ is <u>analytic</u> on G, i.e. f has a power series expansion around each point in G.

$\boxed{EX.1}$ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \leq 0. \end{cases}$

$f \in C^{\infty}(\mathbb{R})$, but is not real analytic on $\mathbb{R}$

(the Taylor series of f around 0 is not equal to f in any neighborhood of 0).

<u>Classical power series expansions (around 0)</u>

1) $\frac{1}{1-z} = 1 + z + z^2 + \ldots + z^n + \ldots$, $z \in U(0,1)$

2) $e^z = 1 + \frac{z}{1!} + \ldots + \frac{z^n}{n!} + \ldots$, $z \in \mathbb{C}$

3) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + \frac{(-1)^n}{(2n)!} z^{2n} + \ldots$, $z \in \mathbb{C}$

4) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + \frac{(-1)^n}{(2n+1)!} z^{2n+1} + \ldots$, $z \in \mathbb{C}$

5) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots + \frac{(-1)^{n+1}}{n} z^n + \ldots$, $z \in U(0,1)$.