Zeros of holomorphic functions and the identity theorem

$\boxed{D1}$ Let $G \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(G)$ and $z_0 \in G$.

z_0 is a zero of f, if $f(z_0)=0$.

z_0 is a zero of order $n \in \mathbb{N}^*$ of f, if $f(z_0)=f'(z_0)=\dots=f^{(n-1)}(z_0)=0$,
$f^{(n)}(z_0)\neq 0$.

$\boxed{T1}$ (factorization of holomorphic functions)

Let $G \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(G)$, $z_0 \in G$ and $n \in \mathbb{N}^*$.

Then: z_0 is a zero of order n of $f \Longleftrightarrow$

$\Longleftrightarrow \exists\, \varphi \in \mathcal{H}(G)$ s.t. $\varphi(z_0)\neq 0$ and $f(z)=(z-z_0)^n\varphi(z)$, $z\in G$.

Proof:

"$\Leftarrow$" If $\varphi \in \mathcal{H}(G)$ s.t. $\varphi(z_0)\neq 0$, $f(z)=(z-z_0)^n\varphi(z)$, $z \in G$,
then $f^{(k)}(z_0)=0$, $k=\overline{0,n-1}$, $f^{(n)}(z_0)=n!\,\varphi(z_0)\neq 0$,
so, z_0 is a zero of order n of f.

"$\Rightarrow$" If $f(z_0)=f'(z_0)=\dots=f^{(n-1)}(z_0)=0$, $f^{(n)}(z_0)\neq 0$,
then $f(z) \underset{\text{T3, Course11}}{=} \sum_{k=n}^{\infty} \underbrace{a_k}_{\frac{f^{(k)}(z_0)}{k!}}(z-z_0)^k$, $z \in U(z_0,r)$, where $r>0$ s.t. $U(z_0,r)\subseteq G$,

so $f(z) \underset{(1)}{=} (z-z_0)^n \sum_{k=n}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-n}$, $z \in U(z_0,r)$.

Let $\varphi: G \to \mathbb{C}$, $\varphi(z)=\begin{cases} \frac{f(z)}{(z-z_0)^n}, & z \in G\setminus\{z_0\} \\ \frac{f^{(n)}(z_0)}{n!}, & z=z_0. \end{cases}$

$\varphi \in \mathcal{H}(G\setminus\{z_0\})$, $(1) \Rightarrow \lim_{z\to z_0}\varphi(z) = \lim_{z\to z_0}\sum_{k=n}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-n} = \frac{f^{(n)}(z_0)}{n!} = \varphi(z_0)$

$\Rightarrow \varphi$ is continuous on $G \xRightarrow[\text{Course11}]{\boxed{C4}} \varphi \in \mathcal{H}(G)$, $\varphi(z_0)\neq 0$, $f(z)=(z-z_0)^n\varphi(z)$,
$z\in G$.

$\boxed{T2}$ (Identity theorem)

Let $D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$. Then the following are equivalent:

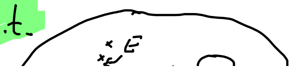
i) $f \equiv 0$.

ii) $\exists\, a \in D$ s.t. $f^{(n)}(a)=0$, $\forall n \in \mathbb{N}$.

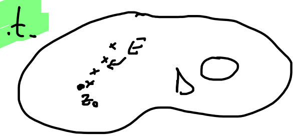
iii) $\exists\, E \subset D$ s.t. $E' \cap D \neq \emptyset$ and $f|_E \equiv 0$.

↳ the set of accumulation/limit points of E

$(E'\cap D \neq \emptyset \Longleftrightarrow$ $\exists\, z_0 \in D$, $\exists (z_k)_{k\in\mathbb{N}^*}$ in $E\setminus\{z_0\}$ s.t.



$(E' \cap D \neq \emptyset \Leftrightarrow \exists z_0 \in D, \exists (z_k)_{k\in\mathbb{N}^*}$ in $E \setminus \{z_0\}$ s.t. $\lim_{k\to\infty} z_k = z_0)$



<u>Proof</u>: Clearly, i) $\Rightarrow$ ii) and i) $\Rightarrow$ iii).

$\boxed{\text{ii)} \Rightarrow \text{i)}}$ Let $A = \{z \in D : f^{(n)}(z) = 0, \forall n \in \mathbb{N}\} \neq \emptyset$.

• <u>A is open in D</u>: $a \in A \subset D \Rightarrow \exists r > 0$ s.t. $U(a,r) \subseteq D$

$$\left.\begin{array}{l} \xrightarrow[f\in\mathcal{H}(U(a,r))]{\text{T3, Course 11}} f(z) = \sum_{n=0}^{\infty} \underbrace{a_n}_{\frac{f^{(n)}(a)}{n!}} (z-a)^n, \ z \in U(a,r) \\ a \in A \Rightarrow a_n = 0, \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow f|_{U(a,r)} \equiv 0 \Rightarrow U(a,r) \subseteq A.$$

• <u>A is closed in D</u>: Let $z_0 \in D$ and $(z_k)_{k\in\mathbb{N}^*}$ in A s.t. $\lim_{k\to\infty} z_k = z_0$.

C1, Course 11 $\Rightarrow f^{(n)} \in \mathcal{H}(D), \forall n \in \mathbb{N} \Rightarrow \lim_{k\to\infty} \underbrace{f^{(n)}(z_k)}_{=0,\ z_k\in A} = f^{(n)}(z_0), \forall n \in \mathbb{N}$

$\Rightarrow f^{(n)}(z_0) = 0, \forall n \in \mathbb{N} \Rightarrow z_0 \in A$.

A is open + closed in D $\xRightarrow{D \text{ is connected}}$ $A = D$ or $A = \emptyset$

$\xRightarrow{A\neq\emptyset} A = D \Rightarrow f \equiv 0$ on D.

$\boxed{\text{iii)} \Rightarrow \text{ii)}}$ Let $z_0 \in D$ and $(z_k)_{k\in\mathbb{N}^*}$ in $E \setminus \{z_0\}$ s.t. $\lim_{k\to\infty} z_k = z_0$.

$f|_E \equiv 0 \Rightarrow \lim_{k\to\infty} \underbrace{f(z_k)}_{=0,\ z_k\in E} = f(z_0) \Rightarrow z_0$ is a zero of f.

Assume that ii) doesn't hold $\Rightarrow \exists n \in \mathbb{N}^*$ s.t.

$f(z_0) = f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0, \ f^{(n)}(z_0) \neq 0$.

$\boxed{\text{T1}} \Rightarrow \exists \varphi \in \mathcal{H}(D), \varphi(z_0) \neq 0, \ f(z) = (z-z_0)^n \varphi(z), \ z \in D$.

$z_k \in E \setminus \{z_0\}, \forall k \in \mathbb{N}^* \Rightarrow f(z_k) = 0, \ (z_k - z_0)^n \neq 0, \ \forall k \in \mathbb{N}^*$

$\Rightarrow \varphi(z_k) = 0, \ \forall k \in \mathbb{N}^*$

$\Rightarrow \lim_{k\to\infty} \underbrace{\varphi(z_k)}_{=0} = \varphi(z_0) = 0$, contradiction with $\varphi(z_0) \neq 0$.

$\boxed{\text{C1}}$ Let $D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$ be s.t. $f \not\equiv 0$. Then the zeros of f are isolated in D.

(z_0 is a zero of $f \Rightarrow \exists r > 0$ s.t. $U(z_0,r) \subseteq D$ and $f(z) \neq 0, \forall z \in \dot{U}(z_0,r)$)

$\boxed{\text{C2}}$ Let $D \subseteq \mathbb{C}$ be a domain and $f, g \in \mathcal{H}(D)$.

If $\exists E \subset D$ s.t. $E' \cap D \neq \emptyset$ and $f|_E = g|_E$, then $f = g$ on D.

$\boxed{C3}$ (maximum modulus theorem)

Let $D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$.

If $\exists\, z_0 \in D$ s.t. $|f(z)| \le |f(z_0)|$, $\forall z \in D$, then f is constant on D.

$\boxed{C4}$ Let $D \subseteq \mathbb{C}$ be a bounded domain and $f : \overline{D} \to \mathbb{C}$ be holomorphic on D and continuous on $\overline{D}$. Then:

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

$\boxed{C5}$ Let $D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$.

If $\exists\, z_0 \in D$ s.t. : i) $\operatorname{Re} f(z) \le \operatorname{Re} f(z_0)$, $z \in D$

or

ii) $\operatorname{Re} f(z) \ge \operatorname{Re} f(z_0)$, $z \in D$,

then f is constant on D.

Proof: Consider $g_{\pm}(z) = e^{\pm f(z)}$, $z \in D$.

Laurent series

$\boxed{D2}$ A Laurent series around $z_0 \in \mathbb{C}$ has the form:

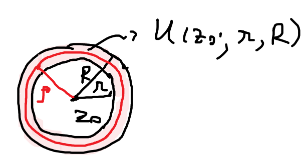
$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \underbrace{\dots + \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z - z_0}}_{\text{the principal part}} +$$

$$\underbrace{+\, a_0 + a_1(z - z_0) + \dots + a_n (z - z_0)^n + \dots}_{\text{the Taylor part}},$$

where $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$, w.r.t. the variable $z \in \mathbb{C} \setminus \{z_0\}$.

$\boxed{T3}$ (Laurent series expansion)

Let $z_0 \in \mathbb{C}$, $0 \le r < R$ and $f \in \mathcal{H}(U(z_0; r, R))$.



Then: $\exists!$ Laurent series around z_0 that converges uniformly on compacts of $U(z_0; r, R)$ (both parts converge u.c.)

s.t. $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, $z \in U(z_0; r, R)$, where

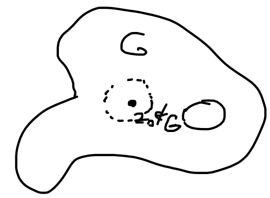
$$a_n = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta, \quad \forall \rho \in (r, R), \ n \in \mathbb{Z},$$

$$\gamma_\rho(t) = z_0 + \rho e^{2\pi i t}, \ t \in [0,1].$$

Isolated singular points

$\boxed{D3}$ Let $f \in \mathcal{H}(G)$, where $G \subseteq \mathbb{C}$ is open.

$z_0 \in \mathbb{C}$ is an isolated singular point of f, if $\exists\, r > 0$ s.t. $\dot{U}(z_0, r) \subseteq G$ and $z_0 \notin G$.



Moreover, we say that z_0 is

i) a removable singular point, if $\underbrace{\exists \lim_{z \to z_0} f(z) \in \mathbb{C}}_{\text{exists and is finite (in } \mathbb{C})}$.

ii) a pole, if $\underbrace{\exists \lim_{z \to z_0} f(z) = \infty}_{\text{exists and is infinite } (\infty \in \mathbb{C}_\infty)}$.

iii) an essential singular point, if $\underbrace{\nexists \lim_{z \to z_0} f(z)}_{\text{doesn't exist in } \mathbb{C}_\infty}$.

Characterization of isolated singular points

Let $z_0 \in \mathbb{C}$, $r > 0$ and $f \in \mathcal{H}(\dot{U}(z_0, r))$.

Consider the Laurent series expansion of f around z_0:

$$(*) \quad f(z) = \dots + \frac{a_{-n}}{(z - z_0)^n} + \dots + \frac{a_{-1}}{z - z_0} +$$

$$+ a_0 + a_1(z - z_0) + \dots + a_n(z - z_0)^n + \dots, \quad z \in \dot{U}(z_0, r)$$

$\boxed{T4}$ i) z_0 is removable $\Leftrightarrow$ $a_{-n} = 0$, $\forall n \in \mathbb{N}^*$, in $(*)$.

ii) z_0 is a pole $\Leftrightarrow$ $\exists\, n \in \mathbb{N}^*$ s.t. $a_{-n} \neq 0$ and $a_{-n-1} = a_{-n-2} = \dots = 0$ ($a_k = 0$, $\forall k < -n$) in $(*)$.

In this case, z_0 is called a pole of order n.

iii) z_0 is essential $\Leftrightarrow$ $\exists$ infinitely many non-zero terms in the principal part of $(*)$.