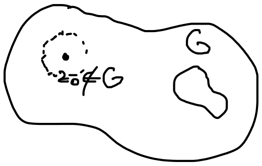
Characterization of isolated singular points (continuation)

Let $z_0\in\mathbb{C}$ to be an isolated singular point for a function $f\in\mathcal{H}(G)$, where $G\subseteq\mathbb{C}$ is open.



$\boxed{T_1}$ The following are equivalent:

i) z_0 is removable.

ii) $\exists F\in\mathcal{H}(G\cup\{z_0\})$ s.t. $F|_G=f$.

iii) $\exists r>0$ s.t. $\dot{U}(z_0,r)\subseteq G$ and f is bounded on $\dot{U}(z_0,r)$.

$\boxed{T_2}$ Let $n\in\mathbb{N}^*$. The following are equivalent:

i) z_0 is a pole of order n.

ii) $\exists g\in\mathcal{H}(U(z_0,r))$, where $\dot{U}(z_0,r)\subseteq G$, s.t. $g(z_0)\neq 0$ and $f(z)=\frac{g(z)}{(z-z_0)^n}$, $z\in\dot{U}(z_0,r)$.

iii) $\exists g,h\in\mathcal{H}(U(z_0,r))$, where $\dot{U}(z_0,r)\subseteq G$, s.t. $g(z_0)\neq 0$, $h(z_0)=h'(z_0)=\dots=h^{(n-1)}(z_0)=0$, $h^{(n)}(z_0)\neq 0$,
$$f(z)=\frac{g(z)}{h(z)},\quad z\in\dot{U}(z_0,r).$$

Calculus of the residue

Let $z_0\in\mathbb{C}$, $r>0$ and $f\in\mathcal{H}(\dot{U}(z_0,r))$. Consider the Laurent series expansion of f around z_0:

$$(*)\quad f(z)=\dots+\frac{a_{-n}}{(z-z_0)^n}+\dots+\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+\dots+a_n(z-z_0)^n+\dots,\quad z\in\dot{U}(z_0,r)$$

$\boxed{D1}$ $\mathrm{Res}(f,z_0)=a_{-1}\underset{\text{Course 12}}{\overset{\boxed{T3}}{=}}\frac{1}{2\pi i}\int_{\gamma_\rho} f(z)\,dz$, $\rho\in(0,r)$, $\gamma_\rho(t)=z_0+\rho e^{2\pi i t}$, $t\in[0,1]$

= is the residue of f at z_0.

$\boxed{R1}$ If z_0 is removable, then $\mathrm{Res}(f,z_0)=0$.

$\boxed{R2}$ If z_0 is a pole of order $n\in\mathbb{N}^*$, then
$$\mathrm{Res}(f,z_0)=\frac{1}{(n-1)!}\lim_{z\to z_0}\left((z-z_0)^n\cdot f(z)\right)^{(n-1)}$$
→ we take here the derivative w.r.t. z $n-1$ times

Proof: z_0 is a pole of ord. $n\in\mathbb{N}^*$ $\underline{\underline{(*)}}$,

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots \quad , z \in \dot{U}(z_0, r)$$

$$\Rightarrow (z-z_0)^n f(z) = a_{-n} + a_{-n+1}\cdot(z-z_0) + \dots + a_{-1}\cdot(z-z_0)^{n-1} + a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + \dots$$

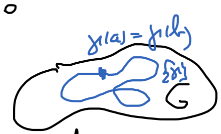
Differentiating $(n-1)$ times (w.r.t. z), we get:

$$\left((z-z_0)^n f(z)\right)^{(n-1)} = (n-1)!\, a_{-1} + \frac{n!}{1!} a_0 (z-z_0) + \frac{(n+1)!}{2!} a_1 (z-z_0)^2 + \dots$$

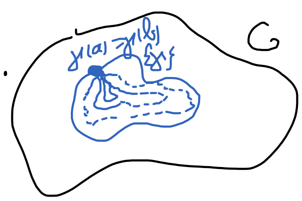
$$\Rightarrow \lim_{z \to z_0} \left((z-z_0)^n f(z)\right)^{(n-1)} = (n-1)!\, \mathrm{Res}(f, z_0).$$

$\boxed{R3}$ If z_0 is an essential isolated singular point, then we find $\mathrm{Res}(f, z_0)$ by expanding f in Laurent series around z_0 and determining a_{-1} = the coefficient of $\frac{1}{z-z_0}$.

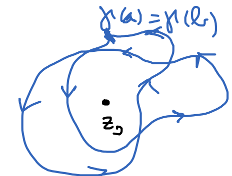
Residue Theorem



$\boxed{D2}$ Let $G \subseteq \mathbb{C}$ be open and $\gamma : [a,b] \to G$ be a contour in G. $(\gamma(a) = \gamma(b))$
γ is null-homotopic in G ($\gamma \underset{G}{\sim} 0$), if
$\exists\ \varphi : [a,b] \times [0,1] \to G$ continuous s.t.
$\varphi(t,0) = \gamma(t)$, $\varphi(t,1) = \underset{=\gamma(b)}{\gamma(a)}$, $t \in [a,b]$, $\varphi(a,s) = \varphi(b,s) = \underset{=\gamma(b)}{\gamma(a)}$, $s \in [0,1]$.

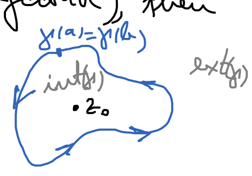


$\boxed{R4}$ If $G \subseteq \mathbb{C}$ is starlike w.r.t. $z_0 \in G$, then for $\forall \gamma$ contour in G: $\gamma \underset{G}{\sim} 0$.



$\boxed{D3}$ Let γ be a contour in $\mathbb{C}$ and $z_0 \in \mathbb{C} \setminus \{\gamma\}$.
$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-z_0}\, dz$ is the index of γ w.r.t. z_0.

$\boxed{R5}$ 1) $n(\gamma, z_0) \in \mathbb{Z}$, $\forall z_0 \in \mathbb{C} \setminus \{\gamma\}$, $\forall \gamma$ contour in $\mathbb{C}$.
2) If γ is a Jordan contour ($\gamma|_{[a,b)}$ is injective), then $\mathbb{C} \setminus \{\gamma\}$ is the union of two domains:
one bounded, denoted by $\mathrm{int}(\gamma)$, and
one unbounded, denoted by $\mathrm{ext}(\gamma)$.



Moreover, if γ traveled/oriented anticlockwise, then $n(\gamma, z_0) = \begin{cases} 1, & z_0 \in \mathrm{int}(\gamma) \\ 0, & z_0 \in \mathrm{ext}(\gamma). \end{cases}$

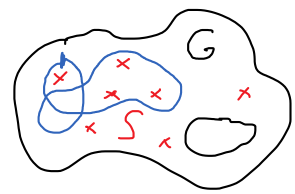
⚠ $n(\gamma^-, z_0) = \begin{cases} -1, & z_0 \in \operatorname{int}(\gamma) \\ 0, & z_0 \in \operatorname{ext}(\gamma). \end{cases}$

↳ the opposite of γ (oriented clockwise)

$\boxed{T_3}$ (Residue Thm.)

Let $G \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(G)$. Let S be the set of isolated singular points of f and $\tilde{G} = G \cup S$.

If γ is a contour in G s.t. $\gamma \underset{\tilde{G}}{\sim} 0$,

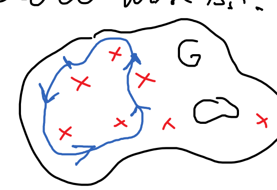


$$\int_\gamma f = 2\pi i \sum_{z_0 \in S} n(\gamma, z_0) \cdot \operatorname{Res}(f, z_0)$$

(where the sum has, in fact, finitely many non-zero terms).

$\boxed{C1}$ Let $G \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(G)$, S be the set of isolated singular points of f and $\tilde{G} = G \cup S$.

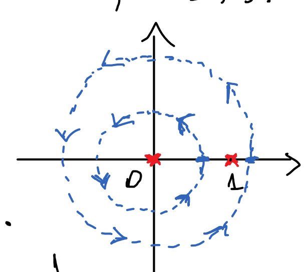
If γ is a Jordan contour oriented anticlockwise s.t. $\gamma \underset{\tilde{G}}{\sim} 0$, then $\int_\gamma f = 2\pi i \sum_{z_0 \in S \cap \operatorname{int}(\gamma)} \operatorname{Res}(f, z_0)$.



$\boxed{\text{Ex.}}$ Compute $J_r = \int_{\gamma_r} \frac{\sin \frac{1}{z}}{(1-z)^2} dz$, $r \in (0, \infty) \setminus \{1\}$, $\gamma_r(t) = r e^{2\pi i t}$, $t \in [0, 1]$.

Sol.: $f : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$

$f(z) = \frac{\sin \frac{1}{z}}{(1-z)^2}$, $z \in \mathbb{C} \setminus \{0, 1\}$.



$z_0 = 0$, $z_1 = 1$ are the isolated sing. points of f.

$\lim_{z \to 0} f(z) = \lim_{z \to 0} \sin \frac{1}{z} \cdot \underbrace{\lim_{z \to 0} \frac{1}{(1-z)^2}}_{1} = \lim_{z \to 0} \sin \frac{1}{z}$,

but this limit doesn't exist: $\lim_{n \to \infty} \sin \frac{1}{\frac{1}{2n\pi}} = 0$,

$\neq$

$\lim_{n \to \infty} \sin \frac{1}{\frac{1}{2n\pi + \frac{\pi}{2}}} = 1$.

So, $z_0 = 0$ is essential.

Since $f(z) = \frac{g(z)}{(z-1)^2}$, $z \in \dot{U}(1, 1)$, where $g(z) = \sin \frac{1}{z}$, $z \in U(1, 1)$, Since $f(z)=\dfrac{g(z)}{(z-1)^2}$, $z\in\dot{U}(1,1)$, where $g(z)=\sin\frac{1}{z}$, $z\in U(1,1)$, $g\in\mathcal{H}(U(1,1))$ and $g(1)=\sin 1\neq 0$, $z_1=1$ is a pole of order 2, by $\boxed{T2}$.

$$J_r \underset{\boxed{R4}}{\overset{\boxed{C1}}{=}} \begin{cases} 2\pi i\,\mathrm{Res}(f,0), & r\in(0,1)\\ 2\pi i\left(\mathrm{Res}(f,0)+\mathrm{Res}(f,1)\right), & r>1. \end{cases}$$

$\mathbb{C}=(\mathbb{C}\setminus\{0,1\})\cup\{z_0,z_1\}$ is starlike $\Rightarrow \gamma_r \underset{\mathbb{C}}{\sim} 0$, $\forall r\in(0,\infty)\setminus\{1\}$.

$\mathrm{Res}(f,0)=?$ $\quad f(z)=\sin\frac{1}{z}\cdot\dfrac{1}{(1-z)^2}=\dots+\dfrac{a_{-n}}{z^n}+\dots+\dfrac{a_{-1}}{z}+a_0+a_1z+\dots$, $z\in\dot{U}(0,1)$

$$\sin y = y-\frac{1}{3!}y^3+\frac{1}{5!}y^5+\dots+(-1)^n\frac{1}{(2n+1)!}y^{2n+1}+\dots,\quad y\in\mathbb{C}$$

$$y=\frac{1}{z}\Rightarrow \sin\frac{1}{z}=\frac{1}{z}-\frac{1}{3!}\cdot\frac{1}{z^3}+\frac{1}{5!}\cdot\frac{1}{z^5}+\dots+(-1)^n\cdot\frac{1}{(2n-1)!}\cdot\frac{1}{z^{2n+1}}+\dots,\quad z\in\dot{U}(0,1)$$

$$\frac{1}{1-z}=1+z+z^2+\dots+z^n+\dots,\quad z\in U(0,1)$$

$$\frac{1}{(1-z)^2}=\left(\frac{1}{1-z}\right)'=1+2z+3z^2+\dots+nz^{n-1}+\dots,\quad z\in\dot{U}(0,1)$$

$\Rightarrow a_{-1}=$ the coef. of $\frac{1}{z}$ in the convolution of the two series

$$=1\cdot 1-\frac{1}{3!}\cdot 3+\dots+\frac{(-1)^n}{(2n+1)!}\cdot(2n+1)+\dots$$

$$=1-\frac{1}{2!}+\dots+\frac{(-1)^n}{(2n)!}+\dots \underset{\text{Course 11}}{=} \cos 1 \Rightarrow \mathrm{Res}(f,0)=\cos 1.$$

$$\mathrm{Res}(f,1)\overset{\boxed{R2}}{=}\frac{1}{(2-1)!}\lim_{z\to 1}\left((z-1)^2\cdot f(z)\right)'=\lim_{z\to 1}\underbrace{\left(\sin\frac{1}{z}\right)'}_{\cos\frac{1}{z}\cdot\left(-\frac{1}{z^2}\right)}=-\cos 1.$$

So, $J_r=\begin{cases}2\pi i\cos 1, & r\in(0,1)\\ 0, & r\in(1,\infty).\end{cases}$