Course14

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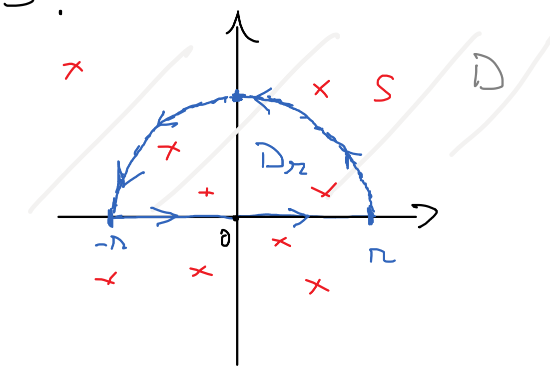
Application of the Residue Theorem for a Fourier integral

$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cdot e^{i\alpha x} dx$, where P, Q are polynomial functions, Q has no zeros on $\mathbb{R}$, $\deg Q > \deg P + 1$, $\alpha \geq 0$.

Let $f : \mathbb{C} \setminus S \to \mathbb{C}$, $S = \{z \in \mathbb{C} : Q(z) = 0\} \subseteq \mathbb{C} \setminus \mathbb{R}$,

$f(z) = \frac{P(z)}{Q(z)} \cdot e^{i\alpha z}$, $z \in \mathbb{C} \setminus S$.

$f \in \mathcal{H}(\mathbb{C} \setminus S)$.



Let $r > 0$, $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

$D_r = D \cap U(0, r)$.

Let $\gamma_1(t) = (-r)(1-t) + rt$, $t \in [0,1]$,

$\gamma_2(t) = r e^{\pi i(t-1)}$, $t \in [1,2]$,

$\gamma_r = \gamma_1 \cup \gamma_2$.

$\boxed{C1}$ (of Residue Thm, Course13) $\Rightarrow \int_{\gamma_r} f = 2\pi i \sum_{z_0 \in S \cap D_r} \operatorname{Res}(f, z_0)$.

$$\int_{\gamma_r} f = \int_{\gamma_1} f + \int_{\gamma_2} f = \int_0^1 f(-r+2rt) \cdot 2r \, dt + \int_1^2 f(r e^{\pi i(t-1)}) \cdot r e^{\pi i (t-1)} \cdot \pi i \, dt$$

$$\xlongequal[\tau = t-1]{x = -r+2rt} \int_{-r}^{r} \frac{P(x)}{Q(x)} \cdot e^{i\alpha x} dx + \int_0^1 \frac{P(r e^{\pi i \tau})}{Q(r e^{\pi i \tau})} \cdot e^{i\alpha r e^{\pi i \tau}} \cdot r e^{\pi i \tau} \cdot \pi i \, d\tau$$

$$\lim_{r \to \infty} \int_{-r}^{r} \frac{P(x)}{Q(x)} \cdot e^{i\alpha} x = I.$$

$|e^{x+iy}| = e^x$

$$\left| \int_0^1 \frac{P(r e^{\pi i \tau})}{Q(r e^{\pi i \tau})} \cdot e^{i\alpha r (\cos(\pi \tau) + i \sin(\pi \tau))} \cdot r \cdot \pi i \cdot e^{\pi i \tau} d\tau \right| \leq$$

$$\leq \int_0^1 \frac{|P(r e^{\pi i \tau})|}{|Q(r e^{\pi i \tau})|} \cdot \underbrace{e^{-\alpha r \overbrace{\sin(\pi \tau)}^{\geq 0}}}_{\leq 1} \cdot r\pi \, d\tau \leq \pi \underbrace{\int_0^1 \frac{r |P(r e^{\pi i \tau})|}{|Q(r e^{\pi i \tau})|} d\tau}_{\downarrow \, r \to \infty}$$

≤ 1

$$\underbrace{\qquad\qquad\qquad}_{\substack{\downarrow\ r\to\infty \\ 0}}$$

because $\deg Q > \deg P + 1$

So, $J = 2\pi i \sum_{z_0 \in D\cap S} \operatorname{Res}(f, z_0)$.

<u>Ex.</u>: $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx = ?$

$$\int_0^\infty \underbrace{\frac{\cos x}{x^2+1}}_{\text{even fct.}}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\frac{e^{ix}+e^{-ix}}{2}}{x^2+1}\,dx \overset{y=-x}{=} \frac{1}{4}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx + \int_{-\infty}^{\infty}\frac{e^{iy}}{y^2+1}\,dy\right)$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx}_{J}.$$

$J = \displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cdot e^{i\alpha x}\,dx$, where $P(x) = 1$, $x \in \mathbb{R}$

$Q(x) = x^2+1$, $x \in \mathbb{R}$

$\alpha = 1$.

$\deg Q = 2 > \underbrace{\deg P}_{=0} + 1 \Rightarrow J = 2\pi i \sum_{z_0 \in S\cap D} \operatorname{Res}(f, z_0)$

$S = \{z \in \mathbb{C} : Q(z) = 0\} = \{z \in \mathbb{C} : z^2+1 = 0\} = \{-i, i\}$

$\Rightarrow J = 2\pi i \operatorname{Res}(f, i)$.

$f(z) = \dfrac{1}{z^2+1}\cdot e^{iz}$, $z \in \mathbb{C}\setminus\{-i, i\}$ $\quad (z^2+1 = (z+i)(z-i))$

$f(z) = \dfrac{g(z)}{z-i}$, $z \in \dot{U}(i,1)$, $g(z) = \dfrac{e^{iz}}{z+i}$, $z \in U(i,1)$,

$g \in \mathcal{H}(U(i,1))$, $g(i) = \dfrac{e^{-1}}{2i} \neq 0 \xRightarrow[\text{C.13}]{\boxed{T_2}}$ i is a pole of ord. 1

$\xRightarrow[\text{C.13}]{\boxed{R_2}} \operatorname{Res}(f, i) = \lim_{z\to i}\underbrace{(z-i)f(z)}_{\frac{e^{iz}}{z+i}} = \dfrac{e^{-1}}{2i} \Rightarrow J = 2\pi i\cdot\dfrac{e^{-1}}{2i} = \dfrac{\pi}{e}$

$\Rightarrow \displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx = \frac{1}{2}J = \frac{\pi}{2e}.$