Course 2

Monday, October 11, 2021 6:30 PM

Topology of the complex plane

Let $A \subseteq \mathbb{C}$.

- A is open $\iff A = \operatorname{int} A$.
- A is closed $\iff A = \overline{A} \iff \mathbb{C} \setminus A$ is open.
- $\overline{A} = A \cup A'$, $\partial A = \overline{A} \setminus \operatorname{int} A$.
- A is compact $\iff A$ is closed and bounded.

Def.: A sequence $(z_n)_{n \in \mathbb{N}}$ in $\mathbb{C}$ converges to $z_0 \in \mathbb{C}$, if $\lim\limits_{n \to \infty} |z_n - z_0| = 0$. Notation: $z_n \to z_0$ or $\lim\limits_{n \to \infty} z_n = z_0$.

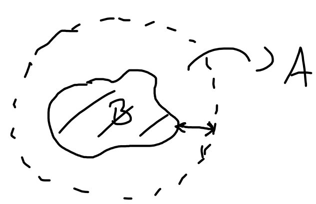
- $z_0 \in \overline{A} \iff \exists (z_n)_{n \in \mathbb{N}}$ in A s.t. $z_n \to z_0$.
- $z_0 \in A' \iff \exists (z_n)_{n \in \mathbb{N}}$ in $A \setminus \{z_0\}$ s.t. $z_n \to z_0$.

(!) If $z_n = x_n + i y_n$, $n \in \mathbb{N}$, $z_0 = x_0 + i y_0$, then:
$\lim\limits_{n \to \infty} z_n = z_0 \iff \lim\limits_{n \to \infty} x_n = x_0$ and $\lim\limits_{n \to \infty} y_n = y_0$.

- $A \subset \mathbb{C}$, $A \neq \emptyset$, is compact $\iff \forall (z_n)_{n \in \mathbb{N}}$ in A, $\exists (z_{n_k})_{k \in \mathbb{N}}$ in $(z_n)_{n \in \mathbb{N}}$ s.t. $\exists \lim\limits_{k \to \infty} z_{n_k} \in A$.

$\boxed{P}$ If $A, B \subseteq \mathbb{C}$ s.t. $A \cap B = \emptyset$, $A \neq \emptyset$ is compact, $B \neq \emptyset$ is closed, then $d(A,B) = \inf\{|a-b| : a \in A, b \in B\} > 0$.

$\boxed{C}$ If $A \subsetneq \mathbb{C}$ is open and $B \subset A$ is compact, $B \neq \emptyset$, then $d(\partial A, B) > 0$.



The induced topology

Def.: Let $A \subseteq B \subseteq \mathbb{C}$.

A is <u>open in B</u>, if $\forall z \in A\ \exists r > 0$ s.t. $U(z,r) \cap B \subseteq A$.

A is <u>closed in B</u>, if $B \setminus A$ is open in B.

(!) A is closed in $B \iff$ [if $(z_n)_{n\in\mathbb{N}}$ in A s.t. $z = \lim_{n\to\infty} z_n \in B$, then $z \in A$].

Connected sets in $\mathbb{C}$

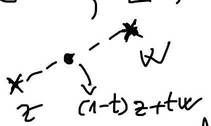
Let $A \subseteq \mathbb{C}$.

Def.: A is <u>connected</u>, if the following holds:
if $B \subseteq A$ is open and closed in A, then $B = \emptyset$ or $B = A$.

(!) A is <u>not</u> <u>connected</u> $\iff \exists A_1, A_2 \subseteq A$ nonempty and open in A s.t. $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$.

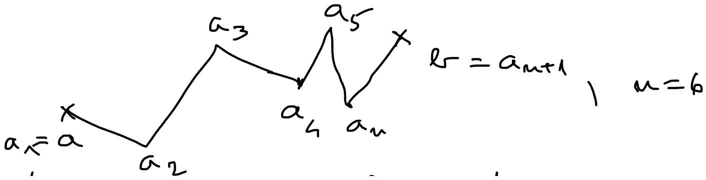
<u>Ex.</u>: If A is the union of two disjoint open disks, then A is <u>not</u> connected.

Notation: for $z, w \in \mathbb{C}$, $[z,w] = \{(1-t)z + tw : t \in [0,1]\}$ is called the <u>segment</u> between z and w.

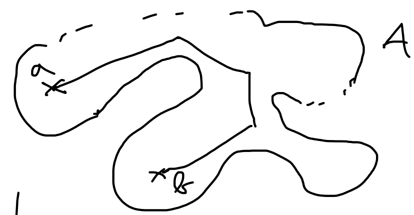


Def.: For $a, b \in \mathbb{C}$, a <u>polygon</u> from a to b is a set

$$\bigcup_{k=1}^{n} [a_k, a_{k+1}], \quad \text{where } a_1 = a,\ a_{n+1} = b,\ a_2, \dots, a_n \in \mathbb{C},\ n \in \mathbb{N}.$$



A is polygonally connected, if $\forall a, b \in A$ there is a polygon from a to b that lies in A.

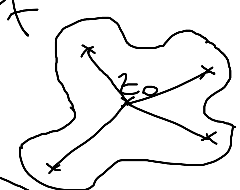


Def.:
A set $D \subseteq \mathbb{C}$ is a domain, if D is open and connected.

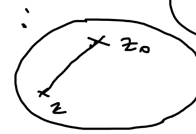
⚠ If $D \subseteq \mathbb{C}$ is open, the D is a domain $\Leftrightarrow$ D is polygonally connected.

Ex.: $\mathbb{C}$, the open disks, the annuli, the punctured disks etc. are domains.

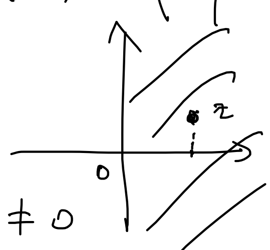
Def.: $D \subseteq \mathbb{C}$ is starlike w.r.t. $z_0 \in D$, if $\forall z \in D$: $[z_0, z] \subset D$.



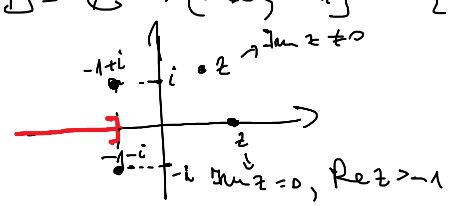
$D \subseteq \mathbb{C}$ is convex, if $\forall z_0 \in D$: D is starlike w.r.t. z_0.



Ex.: • $U(z_0, r)$, $z_0 \in \mathbb{C}$, $r > 0$, is a convex domain.
• $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is the right half-plane, which is a convex domain.



• $D = \mathbb{C} \setminus (-\infty, -1] = \{z \in \mathbb{C} : \operatorname{Im} z \neq 0$ or $\operatorname{Re} z > -1\}$ is a starlike domain, which is not convex.

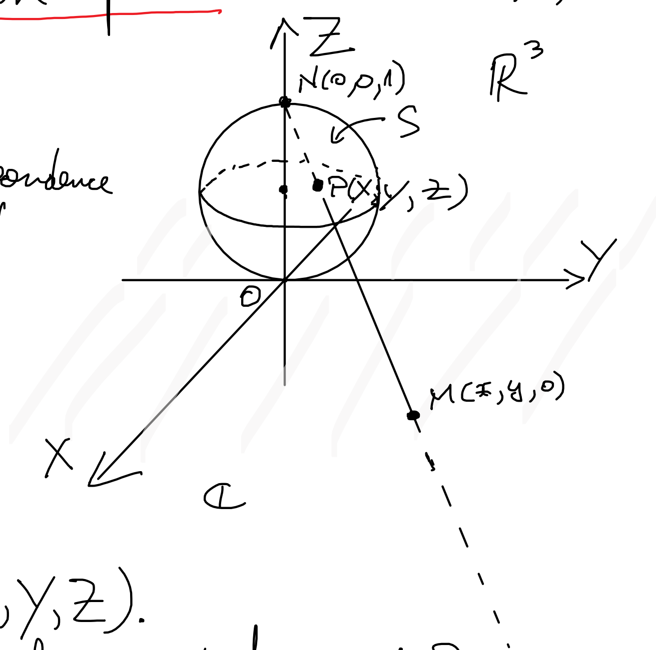


$-1 \pm i \in D$,
but $\underbrace{\frac{1}{2}(-1-i) + \frac{1}{2}(-1+i)}_{-1} \in [-1-i, -1+i]$
$\notin D$

The stereographic projection

We shall consider the one-point compactification of $\mathbb{C}$, by adjoining a point, denoted by ∞, which is not in $\mathbb{C}$, s.t. $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is a compact topological space.
↳ is called the extended complex plane.

Let $S \subset \mathbb{R}^3$ be the sphere: $(S): (X-0)^2 + (Y-0)^2 + \left(Z - \frac{1}{2}\right)^2 = \frac{1}{4}$ and identify the complex plane $\mathbb{C}$ with the XOY plane.
S is called the Riemann sphere. Let $N(0,0,1)$ be the north pole of S.



There is a one-to-one correspondence between the points in $\mathbb{C}$ and the points on $S \setminus \{N\}$:
if $z = x + iy \in \mathbb{C}$ is the affix of $M(x, y, 0)$, then the line NM intersects $S \setminus \{N\}$ at one point $P(X, Y, Z)$.
M is called the stereographic projection of P.
So $\varphi : \mathbb{C} \to S \setminus \{N\}$, given by $\varphi(z) = (X, Y, Z)$, $z = x + iy \in \mathbb{C}$, where z is the affix of stereographic projection of $P(X, Y, Z) \in S \setminus \{N\}$, is bijective.

$\boxed{P_1}$ If $\varphi : \mathbb{C} \to S \setminus \{N\}$ is the above bijection, then: $\varphi(z) = \left(\frac{\operatorname{Re} z}{1 + |z|^2}, \frac{\operatorname{Im} z}{1 + |z|^2}, \frac{|z|^2}{1 + |z|^2} \right)$, $z \in \mathbb{C}$.