Course 3

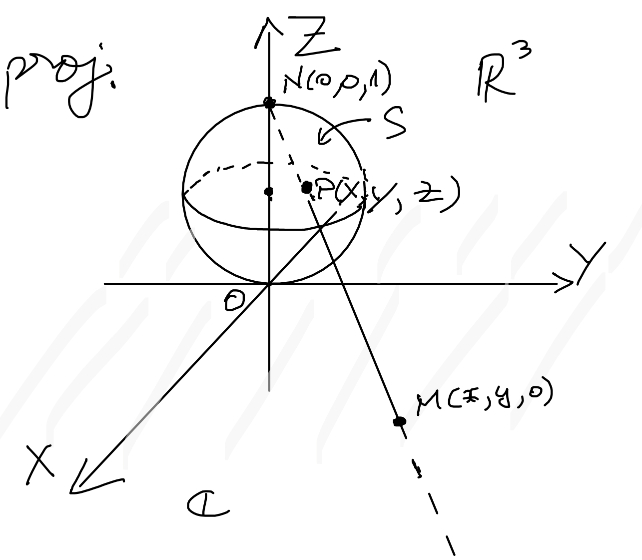
Monday, October 18, 2021 6:33 PM

The topology of the extended complex plane

$\varphi: \mathbb{C} \to S \setminus \{N\}$ is the stereographic projection, where $(S): X^2+Y^2+Z^2-Z=0$ is the Riemann sphere, $N(0,0,1)$ is the north pole.

$$\varphi(z) = \Big(\underbrace{\frac{\operatorname{Re} z}{|z|^2+1}}_{X}, \underbrace{\frac{\operatorname{Im} z}{|z|^2+1}}_{Y}, \underbrace{\frac{|z|^2}{|z|^2+1}}_{Z}\Big), \quad z \in \mathbb{C}.$$

$\{P(X,Y,Z)\} = NM \cap S$, $M(x,y,0) \in XOY$, $z = x+iy$.



We consider the stereographic proj. of the north pole to be the point at infinity:

$$N(0,0,1) \longmapsto \infty \notin \mathbb{C}$$

and thus we obtain the extended complex plane

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

$$\tilde{\varphi}: \mathbb{C}_\infty \to S, \quad \tilde{\varphi}(z) = \begin{cases} \varphi(z), & z \in \mathbb{C} \\ (0,0,1), & z = \infty, \end{cases}$$

is bijective function called the stereogr. proj. of the extended complex plane $\mathbb{C}_\infty$.

Def.: $V \subseteq \mathbb{C}_\infty$ is a neighborhood of ∞, if $\exists r > 0$ s.t. $\{z \in \mathbb{C} : |z| > r\} \cup \{\infty\} \subseteq V$.

$V \subseteq \mathbb{C}_\infty$ is a neighborhood of $z_0 \in \mathbb{C}$, if $\exists r > 0$ s.t. $\{z \in \mathbb{C} : |z - z_0| < r\} \subseteq V$.

Notation: $\mathcal{V}(z)$ the family of neighborhoods of $z \in \mathbb{C}_\infty$.

If $P(X,Y,Z)=\tilde{\varphi}(z)\in S\setminus\{N\}$, $z\in\mathbb{C}$, then the Euclidean distance between N and P is

$$\|NP\|=\sqrt{(X-0)^2+(Y-0)^2+(Z-1)^2}=\sqrt{\underbrace{X^2+Y^2+Z^2}_{Z}-2Z+1}=\sqrt{1-Z}$$

$$=\sqrt{1-\frac{|z|^2}{|z|^2+1}}=\frac{1}{\sqrt{|z|^2+1}}.$$

So, $|z|>r \iff \|NP\|<\frac{1}{\sqrt{r^2+1}}$.

(!) $V\in\mathcal{V}(\infty) \iff \tilde{\varphi}(V)$ is a neighborhood of N on S.

(!) If $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}$, then:

$$\lim_{n\to\infty} z_n=\infty \iff \forall V\in\mathcal{V}(\infty)\ \exists n_V\in\mathbb{N} \text{ s.t. } z_n\in V,\ n\geq n_V$$

$$\iff \forall r>0\ \exists n_r\in\mathbb{N} \text{ s.t. } |z_n|>r,\ n\geq n_r$$

$$\iff \lim_{n\to\infty}\underbrace{|z_n|}_{\in\mathbb{R}}=\infty \iff \lim_{n\to\infty}\frac{1}{|z_n|}=0$$

$$\iff \lim_{n\to\infty}\frac{1}{z_n}=0.$$

Def.: For $z_1,z_2\in\mathbb{C}_\infty$, let $d_c(z_1,z_2)=\|P_1P_2\|$

$$=\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2},$$

where $P_j(X_j,Y_j,Z_j)=\tilde{\varphi}(z_j)$, $j\in\{1,2\}$.

$d_c:\mathbb{C}_\infty\times\mathbb{C}_\infty\to[0,\infty)$ is called the chordal metric on $\mathbb{C}_\infty$.

$$d_c(z_1,z_2)=\begin{cases}\dfrac{|z_1-z_2|}{\sqrt{|z_1|^2+1}\sqrt{|z_2|^2+1}}, & z_1,z_2\in\mathbb{C}\\[2ex] \dfrac{1}{\sqrt{|z|^2+1}}, & z_1=z\in\mathbb{C},\ z_2=\infty \text{ or } z_1=\infty,\ z_2=z\in\mathbb{C}\\[2ex] 0, & z_1=z_2=\infty.\end{cases}$$

(!)
- $\lim_{n\to\infty} z_n=\infty \iff d_c(z_n,\infty)\to 0.$
- $\lim_{n\to\infty} z_n=z\in\mathbb{C} \iff d_c(z_n,z)\to 0.$

(!) $(\mathbb{C}_\infty,d_c)$ is a complete metric space.

Def.: A generalised circle in $\mathbb{C}_\infty$ is either a (usual) circle or a line.

The stereog. proj. of a circle in S is a generalised circle in $\mathbb{C}_\infty$:
- if the circle passes through N, then it's stereographic proj. is a line;
- if the circle is not passing through N, then it's stereogr. proj. is a circle,

So, the lines are gen. circles passing through ∞.

Complex functions of a complex variable

Let $A \subseteq \mathbb{C}$.

Def.: $f : A \to \mathbb{C}$ is called a complex function of a complex variable. We denote:
- $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$, where $u, v : A \to \mathbb{R}$,
- $f = u + iv$, $f(z) = u(z) + i v(z)$, $z \in A$.

Def.: Let $z_0 \in A'$ $(\exists (z_n)_{n\in\mathbb{N}} \subset A \setminus \{z_0\},\ z_n \to z_0)$ and $l \in \mathbb{C}$. f has limit l at z_0 $(\lim_{z\to z_0} f(z) = l)$, if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall z \in A$ with $0 < |z - z_0| < \delta$:
$$|f(z) - l| < \varepsilon.$$

(!) $$\lim_{z\to z_0} f(z) = l \iff \begin{cases} \lim_{z\to z_0} \underbrace{\operatorname{Re} f}_{u}(z) = \operatorname{Re} l \\ \lim_{z\to z_0} \underbrace{\operatorname{Im} f}_{v}(z) = \operatorname{Im} l. \end{cases}$$

Def.: Let $z_0 \in A'$. f has limit ∞ at z_0 $(\lim_{z\to z_0} f(z) = \infty)$, if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall z \in A$ with $0 < |z - z_0| < \delta$:
$$|f(z)| > \varepsilon.$$

(!) $$\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

(!) $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0.$

Def.: Let $\infty \in A'$ ($\exists (z_n)_{n\in\mathbb{N}} \subset A$ s.t. $z_n \to \infty$) and $l \in \mathbb{C}$.

f has limit l at ∞ ($\lim_{z\to\infty} f(z) = l$), if : $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall z \in A$ with $|z| > \delta$: $|f(z) - l| < \varepsilon$.

f has limit ∞ at ∞ ($\lim_{z\to\infty} f(z) = \infty$), if: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall z \in A$ with $|z| > \delta$: $|f(z)| > \varepsilon$.

(!)
- $\lim_{z\to\infty} f(z) = \lim_{z\to 0} f\left(\frac{1}{z}\right).$
- $\lim_{z\to z_0} f(z) = 0 \iff \lim_{z\to z_0} |f(z)| = 0.$
- $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} |f(z)| = \infty.$
- $\lim_{z\to z_0} f(z) = l \in \mathbb{C}_\infty \iff \forall (z_n)_{n\in\mathbb{N}} \subseteq A$ with $\lim_{n\to\infty} z_n = z_0$: $\lim_{n\to\infty} f(z_n) = l.$

Def.: Let $A \subseteq \mathbb{C}$, $z_0 \in A \cap A'$ and $f : A \to \mathbb{C}$.

f is continuous at z_0, if $\exists \lim_{z\to z_0} f(z) = f(z_0)$.

If $z_0 \in A \setminus A'$ (z_0 is isolated), then $f : A \to \mathbb{C}$ is cont. at z_0.

f is continuous on A, if f is cont. at any $z_0 \in A$.

(!) f is cont. at $z_0 \iff u = \operatorname{Re} f$, $v = \operatorname{Im} f$ are cont. at z_0.

(!) The usual operations for cont. real functions hold for continuous complex functions.