Course 4

Monday, October 25, 2021 6:30 PM

Differentiability in $\mathbb{C}$

Def. 1 Let $J \subseteq \mathbb{R}$ be open, $f: J \to \mathbb{C}$, $t_0 \in J$.
f is differentiable at t_0, if $\exists \lim\limits_{t \to t_0} \frac{f(t)-f(t_0)}{t-t_0} := f'(t_0) \in \mathbb{C}$.

(!) • $f = u + iv : J \to \mathbb{C}$ is diff. at t_0 $\iff$ $u, v : J \to \mathbb{R}$ are diff. at t_0.
• $f'(t_0) = u'(t_0) + i v'(t_0)$.

Let $G \subseteq \mathbb{C}$ be open.

Def. 2 $f: G \to \mathbb{C}$, $z_0 \in G$. f is differentiable at z_0, if

$$\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0) \in \mathbb{C}$$

(called the derivative of f at z_0).

Def. 3 $f: G \to \mathbb{C}$, $z_0 \in G$. f is $\mathbb{C}$-differentiable at z_0, if
$\exists \alpha \in \mathbb{C}$, $\exists \omega : G \setminus \{z_0\} \to \mathbb{C}$ s.t. $\lim\limits_{z \to z_0} \omega(z) = 0$,

$$f(z) = f(z_0) + \alpha(z - z_0) + \omega(z)|z - z_0|, \quad z \in G \setminus \{z_0\}.$$

$\boxed{P_1}$ Let $f: G \to \mathbb{C}$, $z_0 \in G$. Then: f is $\mathbb{C}$-diff. at $z_0 \iff$ f is diff. at z_0.

Proof: f is $\mathbb{C}$-diff. at $z_0 \iff \exists \alpha \in \mathbb{C}$ s.t.

$$\lim_{z \to z_0} \underbrace{\left| \underbrace{\frac{f(z) - f(z_0) - \alpha(z - z_0)}{|z - z_0|}}_{\omega(z)} \right|}_{\frac{|f(z) - f(z_0) - \alpha(z - z_0)|}{|z - z_0|} = \left| \frac{f(z) - f(z_0)}{z - z_0} - \alpha \right|} = 0$$

$$\left(\left| \frac{u}{|w|} \right| = \frac{|u|}{|w|} = \left| \frac{u}{w} \right|, \ \forall u, w \in \mathbb{C},\ w \neq 0 \right)$$

$\iff \exists \alpha \in \mathbb{C}$ s.t. $\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \alpha \iff f$ is diff. at z_0.

(!) f is $\mathbb{C}$-diff. at $z_0 \implies \alpha = f'(z_0)$. (1)

Def. 4 $f = u + iv : G \to \mathbb{C}$, $z_0 = x_0 + i y_0 \in G$.
f is $\mathbb{R}$-differentiable at z_0, if u, v are Fréchet differentiable at (x_0, y_0).

Recall: • $\|(x,y)\| = \sqrt{x^2+y^2}$, $(x,y)\in\mathbb{C}$ (the Euclidean norm in $\mathbb{R}^2$).

• f is $\mathbb{R}$-diff. at $z_0=(x_0,y_0) \Longleftrightarrow$

$$\Longleftrightarrow \begin{cases} \exists\, a_1,b_1\in\mathbb{R},\ \exists\, \omega_1: G\setminus\{(x_0,y_0)\}\to\mathbb{R} \text{ s.t. } \lim\limits_{(x,y)\to(x_0,y_0)} \omega_1(x,y)=0, \\ u(x,y) = u(x_0,y_0) + a_1(x-x_0) + b_1(y-y_0) + \omega_1(x,y)\|(x-x_0,y-y_0)\|,\ (x,y)\in G\setminus\{(x_0,y_0)\} \\ \text{and} \\ \exists\, a_2,b_2\in\mathbb{R},\ \exists\, \omega_2: G\setminus\{(x_0,y_0)\}\to\mathbb{R} \text{ s.t. } \lim\limits_{(x,y)\to(x_0,y_0)} \omega_2(x,y)=0, \\ v(x,y) = v(x_0,y_0) + a_2(x-x_0) + b_2(y-y_0) + \omega_2(x,y)\|(x-x_0,y-y_0)\|,\ (x,y)\in G\setminus\{(x_0,y_0)\}. \end{cases}$$

Remark 1: • $a_1 = \frac{\partial u}{\partial x}(x_0,y_0)$, $a_2 = \frac{\partial v}{\partial x}(x_0,y_0)$, $b_1 = \frac{\partial u}{\partial y}(x_0,y_0)$, $b_2 = \frac{\partial v}{\partial y}(x_0,y_0)$. (2)

• $u,v\in C^1(G)$ (u,v have continuous partial derivatives) $\Rightarrow$ f is $\mathbb{R}$-diff. on G.

$\boxed{P_2}$ Let $f=u+iv: G\to\mathbb{C}$, $z_0=x_0+iy_0\in G$. f is $\mathbb{R}$-diff. at $z_0 \Longleftrightarrow$

$\Longleftrightarrow \exists\, \alpha,\beta\in\mathbb{C}$, $\exists\, \omega: G\setminus\{z_0\}\to\mathbb{C}$ s.t. $\lim\limits_{z\to z_0}\omega(z)=0$,

$$f(z) = f(z_0) + \alpha(x-x_0) + \beta(y-y_0) + \omega(z)|z-z_0|,\quad z=x+iy\in G\setminus\{z_0\}.$$

Proof: $f=u+iv$ is $\mathbb{R}$-diff. at $z_0 \Longleftrightarrow \exists\, a_1,b_1,a_2,b_2\in\mathbb{R}$, $\exists\, \omega_1,\omega_2: G\setminus\{z_0\}\to\mathbb{R}$ s.t. $\lim\limits_{(x,y)\to(x_0,y_0)}\omega_1(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)}\omega_2(x,y)=0$ and

$$u(x,y)+iv(x,y) = u(x_0,y_0)+iv(x_0,y_0) + (a_1+ia_2)(x-x_0) + (b_1+ib_2)(y-y_0) +$$
$$+ (\omega_1(x,y)+i\omega_2(x,y))\|(x-x_0,y-y_0)\|,\ x+iy\in G\setminus\{z_0\}$$

$\alpha=a_1+ia_2$, $\beta=b_1+ib_2$, $\omega(z)=\omega_1(z)+i\omega_2(z)$, $f(z)=u(z)+iv(z)$

$$\Longleftrightarrow\ f(z) = f(z_0) + \alpha(x-x_0) + \beta(y-y_0) + \omega(z)|z-z_0|,\ z=x+iy\in G\setminus\{z_0\}.$$

(!) $\alpha = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$, $\beta = \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)$. (2')

The Cauchy-Riemann theorem

$\boxed{T}$ (the characterization of complex differentiable functions)

Let $G\subseteq\mathbb{C}$ be open, $f=u+iv: G\to\mathbb{C}$, $z_0=x_0+iy_0\in G$.

Then: f is differentiable at $z_0 \Longleftrightarrow$

$\Longleftrightarrow$ f is $\mathbb{R}$-differentiable at z_0 and

$$(*)\ \begin{cases} \frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \\ \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \end{cases}$$

$(*)$ is called the Cauchy-Riemann system (conditions) of $f=u+iv$ at z_0.

Proof: $\boxed{\Rightarrow}$ Assume that f is diff. at z_0.

$\boxed{P_1}$ with (1) $\Rightarrow \exists\, \alpha = f'(z_0)$, $\exists\, \omega : G\setminus\{z_0\} \to \mathbb{C}$ s.t. $\lim\limits_{z\to z_0} \omega(z) = 0$

and $f(z) = f(z_0) + \alpha(z - z_0) + \omega(z)|z - z_0|$, $z \in G\setminus\{z_0\}$

$$= f(z_0) + \alpha(x - x_0) + \underbrace{i\alpha}_{\beta}(y - y_0) + \omega(z)|z - z_0|.$$

$\boxed{P_2}$ with (2') $\Rightarrow$ f is $\mathbb{R}$-diff. at z_0 and

$$\left.\begin{aligned} \alpha &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) \,\Big|\cdot i \\ i\alpha = \beta &= \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0) \end{aligned}\right\} \Rightarrow i\frac{\partial u}{\partial x}(x_0,y_0) - \frac{\partial v}{\partial x}(x_0,y_0) = i\frac{\partial v}{\partial y}(x_0,y_0) + \frac{\partial u}{\partial y}(x_0,y_0) \Rightarrow (*).$$

$\boxed{\Leftarrow}$ Assume that f is $\mathbb{R}$-diff. at z_0 and $(*)$ holds.

$\boxed{P_2}$ with (2') $\Rightarrow \exists\, \alpha = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$, $\exists\, \beta = \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)$,

$\exists\, \omega : G\setminus\{z_0\} \to \mathbb{C}$ s.t. $\lim\limits_{z\to z_0} \omega(z) = 0$ and

$$f(z) = f(z_0) + \alpha(x - x_0) + \beta(y - y_0) + \omega(z)|z - z_0|,\quad z = x+iy \in G\setminus\{z_0\}$$

$$(*) \Rightarrow \beta = -\frac{\partial v}{\partial x}(x_0,y_0) + i\frac{\partial u}{\partial x}(x_0,y_0) = i\Big(\underbrace{\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)}_{\alpha}\Big)$$

$$\beta = i\alpha \Rightarrow f(z) = f(z_0) + \alpha(x - x_0) + i\alpha(y - y_0) + \omega(z)|z - z_0|,\ z = x+iy \in G\setminus\{z_0\}$$
$$= f(z_0) + \alpha(z - z_0) + \omega(z)|z - z_0|$$

$\Rightarrow f$ is $\mathbb{C}$-diff. at $z_0 \overset{\boxed{P_1}}{\Longrightarrow} f$ is diff. at z_0.

(!) All the conditions in the C.-R. theorem are essential.

Ex.1 $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$, $z \in \mathbb{C}$.

$f = u + iv \Rightarrow u(x,y) = x$,
$v(x,y) = -y$, $z = x+iy \in \mathbb{C}$.

f is $\mathbb{R}$-diff. on $\mathbb{C}$ (see Remark 1), but
f is not diff. at any $z = x+iy \in \mathbb{C}$, because

$$\frac{\partial u}{\partial x}(x,y) = 1 \neq \frac{\partial v}{\partial y}(x,y) = -1 \Rightarrow (*) \text{ does } \underline{\text{not}} \text{ hold.}$$

Ex.2 $f:\mathbb{C}\to\mathbb{C}$, $f(z)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}+i\dfrac{xy}{\sqrt{x^2+y^2}}, & z=x+iy\in\mathbb{C}^*\\ 0, & z=0\end{cases}$,

is cont. on $\mathbb{C}$, $f=u+iv$, u,v have partial derivatives and satisfy the C.-R. sys. at $z=0$, but f is not diff. at $z=0$.