Course 5

Applications of the Cauchy-Riemann theorem

$\boxed{T}$ (the characterization of complex differentiable functions)

Let $G \subseteq \mathbb{C}$ be open, $f = u + iv : G \to \mathbb{C}$, $z_0 = x_0 + iy_0 \in G$.

Then: f is differentiable at $z_0 \iff$

$\iff$ f is $\mathbb{R}$-differentiable at z_0 and

$$(*) \begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \end{cases}$$

$(*)$ is called the Cauchy-Riemann system (conditions) of $f = u + iv$ at z_0.

Notations: If $f = u + iv : G \to \mathbb{C}$ is $\mathbb{R}$-differentiable at $z_0 = x_0 + iy_0 \in G$, then we denote:

$$\frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$$

$$\frac{\partial f}{\partial y}(z_0) = \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0).$$

Remark 1 From the proof of the C.-R. th., we have: if $f : G \to \mathbb{C}$ is differentiable at $z_0 \in G$, then

$$\underset{-i\beta}{\overset{\parallel}{\alpha}} = f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i \frac{\partial f}{\partial y}(z_0).$$

We consider the differentiable operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$

$\boxed{P_1}$ Let $f : G \to \mathbb{C}$ be $\mathbb{R}$-differentiable at $z_0 \in G$. Then: the C.-R. sys. $(*) \iff \frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

Moreover, if f is differentiable at z_0, then $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

Proof: Let $f = u + iv$, $z_0 = x_0 + iy_0$.

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)\right) + i\left(\frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0)\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}(x_0, y_0) - \frac{\partial v}{\partial y}(x_0, y_0)\right) + i\left(\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0)\right)\right].$$

So, $\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \iff (*)$. If f is differentiable at z_0, then

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\Big(\underbrace{\frac{\partial f}{\partial x}(z_0)}_{f'(z_0)} \underbrace{- i \frac{\partial f}{\partial y}(z_0)}_{f'(z_0) \text{ (see Remark 1)}}\Big) = f'(z_0).$$

$\boxed{P_2}$ If $f: G \to \mathbb{C}$ is $\mathbb{R}$-diff. at $z_0 \in G$, then $\exists\, \omega : G \setminus \{z_0\} \to \mathbb{C}$ s.t. $\lim_{z \to z_0} \omega(z) = 0$ and

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0) \cdot (z - z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \overline{(z - z_0)} + \omega(z)\, |z - z_0|, \quad z \in G \setminus \{z_0\}.$$

Proof : See $\boxed{P_2}$ in Course 4.

Properties Let $f, g : G \to \mathbb{C}$ be $\mathbb{R}$-diff. at $z_0 \in G$.

i) $\forall \alpha, \beta \in \mathbb{C}$: $\alpha f + \beta g$ is $\mathbb{R}$-diff. at z_0 and

$$\frac{\partial}{\partial z}(\alpha f + \beta g)(z_0) = \alpha \frac{\partial f}{\partial z}(z_0) + \beta \frac{\partial g}{\partial z}(z_0)$$

$$\frac{\partial}{\partial \bar{z}}(\alpha f + \beta g)(z_0) = \alpha \frac{\partial f}{\partial \bar{z}}(z_0) + \beta \frac{\partial g}{\partial \bar{z}}(z_0).$$

ii) $f \cdot g$ is $\mathbb{R}$-diff. at z_0 and

$$\frac{\partial}{\partial z}(f \cdot g)(z_0) = \frac{\partial f}{\partial z}(z_0) \cdot g(z_0) + f(z_0) \cdot \frac{\partial g}{\partial z}(z_0)$$

$$\frac{\partial}{\partial \bar{z}}(f \cdot g)(z_0) = \frac{\partial f}{\partial \bar{z}}(z_0) \cdot g(z_0) + f(z_0) \cdot \frac{\partial g}{\partial \bar{z}}(z_0).$$

iii) If f and g are differentiable at z_0, then

$$(f \cdot g)'(z_0) = f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0)$$

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\, g(z_0) - f(z_0) \cdot g'(z_0)}{g(z_0)^2}, \text{ if } g(z_0) \neq 0.$$

Let $G_1, G_2 \subseteq \mathbb{C}$ be open, $f : G_1 \to G_2$, $g : G_2 \to \mathbb{C}$, $z_0 \in G_1$.

iv) If f and g are differentiable at z_0, resp. $f(z_0)$, then $g \circ f$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0).$$

v) If f is $\mathbb{R}$-diff. at z_0 and g is differentiable at $f(z_0)$, then $g \circ f$ is $\mathbb{R}$-diff. at z_0 and

$$\frac{\partial (g \circ f)}{\partial z}(z_0) = g'(f(z_0)) \cdot \frac{\partial f}{\partial z}(z_0)$$

$$\frac{\partial (g \circ f)}{\partial \bar{z}}(z_0) = g'(f(z_0)) \cdot \frac{\partial f}{\partial \bar{z}}(z_0).$$

Ex.1 i) $\frac{\partial z}{\partial z} = 1$, $\frac{\partial z}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}}{\partial z} = 0$, $\frac{\partial \bar{z}}{\partial \bar{z}} = 1$.

ii) $\frac{\partial z^n}{\partial z} = n z^{n-1}$, $\frac{\partial z^n}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}^n}{\partial \bar{z}} = n \bar{z}^{n-1}$, $\frac{\partial \bar{z}^n}{\partial z} = 0$, $n \in \mathbb{N}^*$.

Ex. 2 $f:\mathbb{C}\to\mathbb{C}$, $f(z)=\underline{Az^2}+\underline{B\bar{z}^2}+C\underbrace{|z|^2}_{z\cdot\bar{z}}+\underline{Dz}+\underline{E\bar{z}}+\underline{F}$, $z\in\mathbb{C}$,
where $A,B,C,D,E,F\in\mathbb{C}$. We want to find all $z_0\in\mathbb{C}$ s.t. f is differentiable at z_0. First, we note that $u=\operatorname{Re}f$, $v=\operatorname{Im}f$, then $u,v\in C^1(\mathbb{R}^2)$ (are pol. functions), so f is $\mathbb{R}$-diff. on $\mathbb{C}$ (see Remark 1 in Course 4).

$$\frac{\partial f}{\partial\bar{z}}(z_0)=0+2B\bar{z}_0+C\cdot z_0\cdot 1+0+E,\ \forall z_0\in\mathbb{C}.$$

So, f is differentiable at z_0 $\overset{\boxed{P_1}}{\Longleftrightarrow}$ $\frac{\partial f}{\partial\bar{z}}(z_0)=0 \Longleftrightarrow 2B\bar{z}_0+Cz_0+E=0$ and

$$f'(z_0)=\frac{\partial f}{\partial z}(z_0)=2Az_0+0+C\cdot\bar{z}_0+D.$$

Def. 1 Let $G\subseteq\mathbb{C}$ open, $f:G\to\mathbb{C}$.
f is holomorphic on G, if f is differentiable at any $z\in G$.

We denote: $\mathcal{H}(G)=\{f:G\to\mathbb{C} : f \text{ is holomorphic on } G\}$.
If $f\in\mathcal{H}(\mathbb{C})$, then f is entire.
We say that f is holomorphic at $z_0\in G$, if $\exists r>0$ s.t. $U(z_0,r)\subset G$ and $f\in\mathcal{H}(U(z_0,r))$.

Ex. 3 $f:\mathbb{C}\to\mathbb{C}$, $f(z)=z\cdot|z|^2$, $z\in\mathbb{C}$.
Then f is differentiable at $z_0=0$, but f is not holomorphic at $z_0=0$.

Proof: f is $\mathbb{R}$-diff. on $\mathbb{C}$ ($\operatorname{Re}f,\operatorname{Im}f\in C^1(\mathbb{R}^2)$)
and $\frac{\partial f}{\partial\bar{z}}(z)=\frac{\partial}{\partial\bar{z}}(z\cdot z\cdot\bar{z})=\frac{\partial}{\partial\bar{z}}(z^2\cdot\bar{z})=z^2=0 \Leftrightarrow z=0$.

So, f is diff. only at $z_0=0$ (see $\boxed{P_1}$)
and $f'(0)=\frac{\partial f}{\partial z}(0)=\frac{\partial}{\partial z}(z^2\cdot\bar{z})\Big|_{z=0}=(2z\bar{z})\Big|_{z=0}=0$,
but $\nexists r>0$ s.t. $f\in\mathcal{H}(U(0,r))$, so f is not holomorphic at $z_0=0$.

Examples of entire functions

① **The complex polynomial function**

$p : \mathbb{C} \to \mathbb{C}$, $p(z) = a_0 + a_1 z + \dots + a_n z^n$, $z \in \mathbb{C}$, where $a_0, \dots, a_n \in \mathbb{C}$. Then $p \in \mathcal{H}(\mathbb{C})$ and

$$p'(z) = a_1 + 2a_2 \cdot z + \dots + n a_n z^{n-1}, \ z \in \mathbb{C}.$$

② **The complex exponential function**

From Seminar 4, we have: if $z = x + iy \in \mathbb{C}$, then

$$\exists \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = e^z, \text{ where } e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

$\exp : \mathbb{C} \to \mathbb{C}^*$, $\exp(z) = e^z$, $z \in \mathbb{C}$, is the complex exponential function.

<u>Remark 2</u> $\exp \in \mathcal{H}(\mathbb{C})$ and $(e^z)' = e^z$, $z \in \mathbb{C}$.

<u>Proof</u>: $\exp = u + iv \Rightarrow$ $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$, $x + iy \in \mathbb{C}$.

So, $u, v \in C^1(\mathbb{R}^2) \Rightarrow \exp$ is $\mathbb{R}$-diff. on $\mathbb{C}$.

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = e^x \cos y = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) = e^x(-\sin y) = -\frac{\partial v}{\partial x}(x,y) \end{cases}$$

$\Rightarrow$ exp satisfies the C.-R. sys. at any $z = x + iy \in \mathbb{C} \Rightarrow \exp \in \mathcal{H}(\mathbb{C})$.

$$(e^z)' \underset{\text{Remark 1}}{=} \frac{\partial}{\partial x}(e^x(\cos y + i \sin y)) = e^x(\cos y + i \sin y) = e^z, \ z = x + iy \in \mathbb{C}.$$

<u>Remark 3</u> • $|e^z| = e^x$, $\arg(e^z) = y \pmod{2\pi}$, $\forall z = x + iy \in \mathbb{C}$.

• $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$, $z_1, z_2 \in \mathbb{C}$. $e^{-z} = \frac{1}{e^z}$, $z \in \mathbb{C}$.

• $e^{z + 2k\pi i} = e^z \cdot \underbrace{e^{2k\pi i}}_{1} = e^z$, $\forall z \in \mathbb{C}$, $k \in \mathbb{Z}$.

So, exp is $(2\pi i)$-periodic on $\mathbb{C}$ ($\Rightarrow$ exp is not injective on $\mathbb{C}$).

• $z = x \in \mathbb{R} \Rightarrow \exp(z) = e^x$ (real exponential fct. at $x \in \mathbb{R}$).

- $\forall z \in \mathbb{C}^*: \quad z = r \cdot e^{i\theta}$, $r = |z|$, $\theta \in \operatorname{Arg} z$.
(trig. form)

- $\begin{cases} e^{iy} = e^0(\cos y + i \sin y) \\ e^{-iy} = e^0(\cos y - i \sin y) \end{cases} \Rightarrow \begin{cases} \cos y = \dfrac{e^{iy} + e^{-iy}}{2} \\ \sin y = \dfrac{e^{iy} - e^{-iy}}{2i} \end{cases}$ (Euler's formulae), $y \in \mathbb{R}$.

③ The complex trigonometric functions

$\cos, \sin : \mathbb{C} \to \mathbb{C}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $z \in \mathbb{C}$

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $z \in \mathbb{C}$.

Remarks 3
- $\cos, \sin \in \mathcal{H}(\mathbb{C})$
  $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$, $z \in \mathbb{C}$.
- $\cos(-z) = \cos z$, $\sin(-z) = -\sin z$, $\forall z \in \mathbb{C}$.
- $\cos(z + 2k\pi) = \cos z$, $\sin(z + 2k\pi) = \sin z$, $\forall z \in \mathbb{C}, \forall k \in \mathbb{Z}$.
- $\cos^2 z + \sin^2 z = 1$, $\forall z \in \mathbb{C}$.

④ The complex hyperbolic functions

$\operatorname{ch}, \operatorname{sh} : \mathbb{C} \to \mathbb{C}$, $\operatorname{ch} z = \dfrac{e^z + e^{-z}}{2}$, $\operatorname{sh} z = \dfrac{e^z - e^{-z}}{2}$, $z \in \mathbb{C}$.

Remarks 4
- $\operatorname{ch}, \operatorname{sh} \in \mathcal{H}(\mathbb{C})$, $(\operatorname{ch} z)' = \operatorname{sh} z$
  $(\operatorname{sh} z)' = \operatorname{ch} z$, $z \in \mathbb{C}$.
- $\operatorname{ch}^2 z - \operatorname{sh}^2 z = 1$, $\forall z \in \mathbb{C}$.