Course 6

Monday, November 8, 2021 6:30 PM

A criterion for holomorphic functions to be constant

Recall: $D \subseteq \mathbb{C}$ is a domain, if D is open and connected.

(!) $f \in \mathcal{H}(G)$, $G \subseteq \mathbb{C}$ open $\Rightarrow$ $f \in C(G)$ (continuous on G).

$\boxed{T_1}$ Let $\emptyset \neq D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$, $f = u + iv$.

The the following are equivalent: i) f is constant on D.

ii) $f' \equiv 0$ on D.

iii) $\operatorname{Re} f$ is constant on D.

iv) $\operatorname{Im} f$ is constant on D.

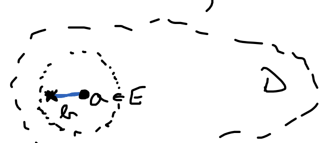
v) $|f|$ is constant on D.

Proof: $\boxed{\text{i)} \Rightarrow \text{ii), iii), iv), v)}}$ is clear.

$\boxed{\text{ii)} \Rightarrow \text{i)}}$ Assume that $f' \equiv 0$ on D. Fix $z_0 \in D$ and $E = \{z \in D : f(z) = f(z_0)\}$.

- E is closed in D: if $(z_n)_{n \in \mathbb{N}}$ is in E s.t. $z = \lim_{n \to \infty} z_n \in D$, then $z \in E$, because $f \in \mathcal{H}(D) \Rightarrow f \in C(D) \Rightarrow \lim_{n \to \infty} \underbrace{f(z_n)}_{f(z_0)} = f(z)$
$\Rightarrow f(z_0) = f(z) \Rightarrow z \in E$.

- E is open in D: Let $a \in E$. Since $a \in D$, $\exists r > 0$ s.t. $U(a, r) \subseteq D$. Let $b \in U(a, r)$.



Then $[a, b] = \{(1-t)a + tb : t \in [0,1]\} \subset U(a, r)$.

Let $\gamma : [0,1] \to D$, $\gamma(t) = (1-t)a + tb$, $t \in [0,1]$.

γ is differentiable at any $t_0 \in (0,1)$ and $\gamma([0,1]) = [a, b]$.

Let $g = f \circ \gamma$. For $t, t_0 \in [0,1]$, $t \neq t_0$:

$$\frac{g(t) - g(t_0)}{t - t_0} = \frac{f(\gamma(t)) - f(\gamma(t_0))}{\gamma(t) - \gamma(t_0)} \cdot \underbrace{\frac{\gamma(t) - \gamma(t_0)}{t - t_0}}_{(b-a)}.$$

So, $\lim_{t \to t_0} \frac{g(t) - g(t_0)}{t - t_0} \overset{f \in \mathcal{H}(D)}{\underset{\lim_{t \to t_0} \gamma(t) = \gamma(t_0)}{=}} \underbrace{f'(\gamma(t_0))}_{=0} \cdot (b - a) = 0$, $\forall t_0 \in (0,1)$

$\Rightarrow g' \equiv 0$ on $(0,1)$ $\Rightarrow$ g is constant on $[0,1]$

$\left.\begin{array}{l} \Rightarrow g(0) = g(1) \Rightarrow f(a) = f(b) \\ a \in E \Rightarrow f(a) = f(z_0) \end{array}\right\} \Rightarrow f(b) = f(z_0) \Rightarrow b \in E.$

So, $U(a,r) \subseteq E$. We deduce that E is open in D.

E is open and closed in $D \overset{D \text{ connected}}{\Longrightarrow} \left.\begin{array}{l} E = D \text{ or } E = \emptyset \\ z_0 \in E \Rightarrow E \neq \emptyset \end{array}\right\} \Rightarrow$

$\Rightarrow E = D \Rightarrow f$ is constant on D.

$\boxed{iii) \Rightarrow i)}$ Assume that $u = \operatorname{Re} f$ is constant on D.

Then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \equiv 0$ on D.

$f \in \mathcal{H}(D) \xrightarrow[\text{Riemann}]{\text{Cauchy-}} \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \equiv 0$ on D

$\Rightarrow f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv 0$ on $D \overset{ii) \Rightarrow i)}{\Longrightarrow} f$ is constant on D.

$\boxed{iv) \Rightarrow i)}$ is proved as above.

$\boxed{v) \Rightarrow i)}$ Assume that $|f|$ is constant on D. Then $\exists c \geq 0$ s.t. $u^2 + v^2 = c$ on D.

- $c = 0 \Rightarrow u = v \equiv 0 \Rightarrow f \equiv 0$.
- $c > 0$. Let $z = x + iy \in D$.

$u^2 + v^2 = c \Longrightarrow \frac{\partial}{\partial x}(u^2+v^2)(x,y) = \frac{\partial}{\partial y}(u^2+v^2)(x,y) = 0$

$$\Rightarrow \begin{cases} 2u(x,y)\frac{\partial u}{\partial x}(x,y) + 2v(x,y)\frac{\partial v}{\partial x}(x,y) = 0 \\ 2u(x,y)\frac{\partial u}{\partial y}(x,y) + 2v(x,y)\frac{\partial v}{\partial y}(x,y) = 0 \end{cases} \overset{(C.-R.)}{\Longrightarrow} \begin{cases} u(x,y)\frac{\partial u}{\partial x}(x,y) + v(x,y)\frac{\partial v}{\partial x}(x,y) = 0 \\ v(x,y)\frac{\partial u}{\partial x}(x,y) + (-u(x,y))\frac{\partial v}{\partial x}(x,y) = 0 \end{cases}$$

$$\begin{vmatrix} u(x,y) & v(x,y) \\ v(x,y) & -u(x,y) \end{vmatrix} = -u^2(x,y) - v^2(x,y) = -c \neq 0 \Rightarrow$$

$\Rightarrow$ the above lin. homogeneous sys. has a unique sol. $\left(\frac{\partial u}{\partial x}(x,y), \frac{\partial v}{\partial x}(x,y)\right) = (0,0)$.

So, $f' = \frac{\partial f}{\partial x} \equiv 0 \overset{ii) \Rightarrow i)}{\Longrightarrow} f$ is constant on D.

$\boxed{C1}$ Let $D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$.

If $f(D) \subseteq \mathbb{R}$ or $f(D) \subseteq i\mathbb{R}$, then f is constant on D.

Proof: $\left.\begin{array}{l} f(D) \subseteq \mathbb{R} \Rightarrow \operatorname{Im} f \equiv 0 \\ \quad \text{or} \\ f(D) \subseteq i\mathbb{R} \Rightarrow \operatorname{Re} f \equiv 0 \end{array}\right\} \overset{T1}{\Longrightarrow} f$ is constant on D.

Harmonic functions

(!) $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator in $\mathbb{R}^n$.

$\boxed{D1}$ Let $G \subseteq \mathbb{C}$ be open. $u : G \to \mathbb{R}$ is harmonic, if $u \in C^2(G)$ and $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$, $\forall x+iy \in G$

(the Laplace equation)

$\boxed{P_1}$ Let $G \subseteq \mathbb{C}$ open and $f \in \mathcal{H}(G)$, $f = u + iv$.
Then $u = \operatorname{Re} f$, $v = \operatorname{Im} f$ are harmonic on G.

Proof: (!) $f \in \mathcal{H}(G) \Longrightarrow u, v \in C^\infty(G)$ (this will be proved in a future course).

C-R. $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\Rightarrow \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$
$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \overset{v \in C^2(G)}{=\!=} 0 \Rightarrow u \text{ is harmonic.}$$

Similarly, v is harmonic.

Ex.: $u : \mathbb{C} \to \mathbb{R}$, $u(x,y) = e^x \cos y$, $x + iy \in \mathbb{C}$

$\Rightarrow u = \operatorname{Re} f$, where $f(z) = e^z = e^x(\cos y + i \sin y)$, $z = x+iy \in \mathbb{C}$

$\Rightarrow u$ is harmonic on $\mathbb{C}$.

$v = \operatorname{Im} f$ is harmonic on $\mathbb{C}$.

(?) If $G \subseteq \mathbb{C}$ is open, $u : G \to \mathbb{R}$ is harmonic on G, then $\exists f \in \mathcal{H}(G)$ s.t. $u = \operatorname{Re} f$?

$\boxed{D2}$ Let $G \subseteq \mathbb{C}$ be open and $P, Q : G \to \mathbb{R}$ be s.t. $P, Q \in C^1(G)$. Then: $\omega = P\,dx + Q\,dy$ is called a (linear) differential form of class C^1 on G.

$\forall z \in G : \omega(z) = P(z)\,dx + Q(z)\,dy$ is a linear function from $\mathbb{R}^2$ to $\mathbb{R}$, where $dx(x,y) = x$, $dy(x,y) = y$, $(x,y) \in \mathbb{R}^2$.

ω is • closed, if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on G;

• exact, if $\exists\, f : G \to \mathbb{R}$, $f \in C^2(G)$ s.t.

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y} \quad \Big(\text{i.e. } \underbrace{df}_{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy} = \omega \Big).$$