Course 7

Monday, November 15, 2021 6:30 PM

Harmonic functions

$\boxed{P_1}$ (Poincaré) Let $D \subseteq \mathbb{R}^2$ be a starlike domain w.r.t. $z_0 \in D$ and ω be a differential form of class C^1 on D.
Then: ω is closed $\Longleftrightarrow$ ω is exact.

$\boxed{T_1}$ Let $D \subseteq \mathbb{C}$ be a starlike domain w.r.t. $z_0 \in D$ and $u : D \to \mathbb{R}$ be harmonic. Then $\exists\, f \in \mathcal{H}(D)$ s.t. $u = \operatorname{Re} f$.

Proof: We look for $v : D \to \mathbb{R}$, $v \in C^2(D)$ s.t. u and v satisfy the Cauchy-Riemann sys. on D:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

on D, since this implies $f = u + iv \in \mathcal{H}(D)$, by the Cauchy-Riemann thm.

Let $P = -\dfrac{\partial u}{\partial y}$, $Q = \dfrac{\partial u}{\partial x}$ on D.

u is harmonic on $D \Rightarrow u \in C^2(D) \Rightarrow \omega = P\,dx + Q\,dy$ is a diff. form of class C^1 on D.

ω is closed: $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ on $D \Longleftrightarrow \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right)$ on D

$\Longleftrightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on D

$\Longleftrightarrow \Delta u = 0$ on D, which is true because u is harmonic on D.

D is a starlike domain w.r.t. $z_0 \in D$

$\xRightarrow{(P_1)}$ ω is exact $\Rightarrow \exists\, v : D \to \mathbb{R}$, $v \in C^2(D)$ s.t.

$P = \dfrac{\partial v}{\partial x}$ and $Q = \dfrac{\partial v}{\partial y}$ $(\omega = dv)$

$$\Rightarrow \begin{cases} -\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} \quad (*) \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}. \end{cases}$$

$f = u + iv$, $u, v \in C^2(D) \Rightarrow f$ is $\mathbb{R}$-differentiable on D.

$(*) \Rightarrow f$ satisfies the C.-R. sys. on D $\Rightarrow f \in \mathcal{H}(D)$.

$\boxed{C_1}$ Let $D \subseteq \mathbb{C}$ be a starlike domain w.r.t. $z_0 \in D$ and $v : D \to \mathbb{R}$ be harmonic. Then $\exists\, f \in \mathcal{H}(D)$ s.t. $v = \operatorname{Im} f$.

Proof: $\boxed{T_1} \Rightarrow \exists\, g \in \mathcal{H}(D)$ s.t. $v = \operatorname{Re} g$

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Proof: $\boxed{T_1} \Rightarrow \exists\, g \in \mathcal{H}(D)$ s.t. $v = \operatorname{Re} g$.
Let $f = ig$. $f \in \mathcal{H}(D)$ and $\operatorname{Im} f = v$.

The complex multivalued logarithm

Let $w \in \mathbb{C}^*$. The eq. $e^z = w$ has infinitely many solutions: $e^z = w \overset{z = x+iy}{\Longleftrightarrow} e^x(\cos y + i \sin y) = |w|\big(\cos(\arg w) + i \sin(\arg w)\big)$

$$\Longleftrightarrow \begin{cases} e^x = |w| > 0 \quad | \ln \\ y = \arg w + 2k\pi, \ k \in \mathbb{Z} \end{cases} \Longleftrightarrow z_k = \ln|w| + i(\arg w + 2k\pi), \ k \in \mathbb{Z}.$$

$\boxed{D1}$ Let $\text{Log} : \mathbb{C}^* \to \mathcal{P}(\mathbb{C})$,

$$\text{Log}\, z = \{\ln|z| + i(\arg z + 2k\pi) : k \in \mathbb{Z}\} = \ln|z| + i \operatorname{Arg} z, \quad z \in \mathbb{C}^*.$$

Log is the (complex) multivalued logarithm.

$\boxed{R1}$ 1) $w \in \text{Log}\, z \Longleftrightarrow e^w = z \quad (\forall z \in \mathbb{C}^*, w \in \mathbb{C})$.
2) $\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2, \ \forall z_1, z_2 \in \mathbb{C}^*$.
3) $\text{Log}\, \frac{1}{z} = -\text{Log}\, z, \ \forall z \in \mathbb{C}^*$.

$\boxed{D2}$ Let $D \subseteq \mathbb{C}^*$ be a domain. $f : D \to \mathbb{C}$ is a branch of Log on D, if $f \in \mathcal{H}(D)$ and

$$f(z) \in \text{Log}\, z, \ \forall z \in D.$$

$\boxed{P2}$ Let $D \subseteq \mathbb{C}^*$ be a domain.
i) If f is a branch of Log on D, then $f'(z) = \frac{1}{z}, \ \forall z \in D$.
ii) If f_1, f_2 are branches of Log on D, then $\exists\, k_0 \in \mathbb{Z}$ s.t. $f_2(z) = f_1(z) + 2k_0\pi i, \ \forall z \in D$.
Moreover, if, in addition, $\exists\, z_0 \in D$ s.t. $f_1(z_0) = f_2(z_0)$, then $f_1(z) = f_2(z), \ \forall z \in D$.

Proof: i) $f \in \mathcal{H}(D)$ and $e^{f(z)} = z, \ \forall z \in D$

$$\Rightarrow \left(e^{f(z)}\right)' = e^{f(z)} \cdot f'(z) = 1, \ \forall z \in D$$

$$\Rightarrow f'(z) = \frac{1}{z}, \ \forall z \in D.$$

ii) $\forall z \in D: \exists\, \theta_1(z), \theta_2(z) \in \operatorname{Arg} z$ s.t.

$$f_1(z) = \ln|z| + i\theta_1(z)$$
$$f_2(z) = \ln|z| + i\theta_2(z).$$

$\forall z \in D: \exists\, k(z) \in \mathbb{Z}$ s.t. $\theta_2(z) = \theta_1(z) + 2k(z)\pi$.

i) $\Rightarrow f_1'(z) = f_2'(z) = \frac{1}{z}, \ \forall z \in D$

$\Rightarrow (f_2 - f_1)' \equiv 0$ on D, D is domain

$\xRightarrow[f_1, f_2 \in \mathcal{H}(D)]{\boxed{T_1}, \text{Course 6}}$ $f_2 - f_1$ is constant on D

$\Rightarrow \theta_2 - \theta_1$ is constant on D

$\Rightarrow k$ is constant on D

$\Rightarrow \exists\, k_0 \in \mathbb{Z}$ s.t. $f_2(z) = f_1(z) + 2k_0\pi \cdot i, \ \forall z \in D$.

If $\exists\, z_0 \in D$ s.t. $f_1(z_0) = f_2(z_0) \Rightarrow k_0 = 0 \Rightarrow f_2 = f_1$ on D.

$\boxed{P_3}$ Let $B = \{ z \in \mathbb{C} : -\pi < \operatorname{Im} z < \pi \}$,

$D = \mathbb{C} \setminus (-\infty, 0] = \mathbb{C} \setminus \{ w \in \mathbb{C} : \operatorname{Re} w \le 0, \operatorname{Im} w = 0 \}$

and $f: B \to D$, $f(z) = \exp(z) = e^z$, $z \in B$.

Then f is bijective on B and $f^{-1}(w) = \ln|w| + i \arg w$, $w \in D$.

Proof: From Seminar 7, we have:

$B_\alpha = \{ z \in \mathbb{C} : \alpha \le \operatorname{Im} z < \alpha + 2\pi \}, \ \alpha \in \mathbb{R}$,

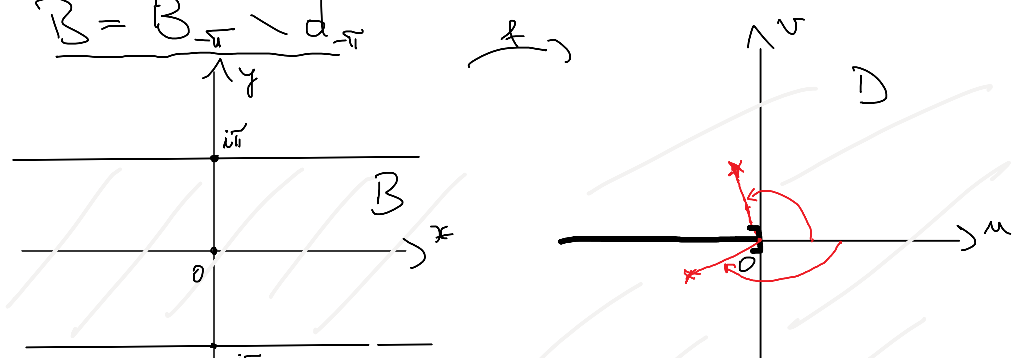
$d_a = \{ z \in \mathbb{C} : \operatorname{Re} z = a \} \parallel Oy, \ a \in \mathbb{R}$

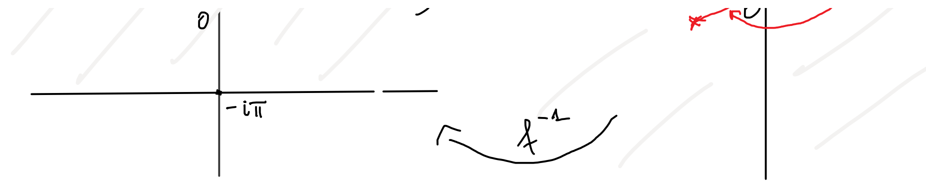
$\tilde{d}_b = \{ z \in \mathbb{C} : \operatorname{Im} z = b \} \parallel Ox, \ b \in \mathbb{R}$,

$\exp$ is injective on B_α, $\exp(B_\alpha) = \mathbb{C}^*$,

$\exp(d_a) = \partial U(0, e^a)$, $\exp(\tilde{d}_b) = \{ w \in \mathbb{C}^* : \arg w = b \ (\text{mod } 2\pi) \}$.

$B = B_{-\pi} \setminus \tilde{d}_{-\pi}$





f is injective on B and $f(B) = \mathbb{C}^* \setminus \exp(d_{-\pi})$

$= \mathbb{C}^* \setminus \{w \in \mathbb{C}^* : \arg w = -\pi \pmod{2\pi}\}$

$= \mathbb{C}^* \setminus (-\infty, 0)$

$= \mathbb{C} \setminus (-\infty, 0] = D.$

Let $w \in D$. $f(z) = w,\ z \in B \iff e^z = w,\ z \in B$

$\iff e^{\operatorname{Re} z}(\cos \operatorname{Im} z + i \sin \operatorname{Im} z) = |w|(\cos(\arg w) + i \sin(\arg w)),\ z \in B$

$\iff \begin{cases} \operatorname{Re} z = \ln|w| > 0 \\ \operatorname{Im} z = \arg w \in (-\pi, \pi) \end{cases} \iff z = \ln|w| + i \arg w.$

So, $f^{-1}(w) = \ln|w| + i \arg w,\ \forall w \in D$.

$\boxed{D_3}$ Let $D = \mathbb{C} \setminus (-\infty, 0]$, $B = \{w \in \mathbb{C} : -\pi < \operatorname{Im} w < \pi\}$.

$\log : D \to B$, $\log z = \ln|z| + i \arg z$, $z \in D$,

is the principal branch of Log on D.

$\boxed{R_2}$ • $\log \in \mathcal{H}(D)$ (see the next seminar) and

$\log 1 = 0 \overset{\boxed{P_2}}{\Longrightarrow} \log$ is the unique branch of Log on D

s.t. $\log 1 = 0$.

• $\boxed{P_2} \Rightarrow (\log z)' = \frac{1}{z},\ z \in D$.

• $z = x \in (0, \infty) \Rightarrow \log x = \ln x$. So, $\log\big|_{(0,\infty)} = \ln$.

$\boxed{R_3}$ • $\forall k \in \mathbb{Z}$: $\log_k : D \to \mathbb{C}$, $\log_k = \log + 2k\pi i$

is the unique branch of Log on D

s.t. $\log_k 1 = 2k\pi i$.

• $\log_k(D) = 2k\pi i + B,\ \forall k \in \mathbb{Z}$.