Course 8

Monday, November 22, 2021 6:30 PM

The geometric interpretation of the complex derivative

Let $G \subseteq \mathbb{C}$ be open, $a, b \in \mathbb{R}$, $a < b$.

$\boxed{D_1}$ Let $\gamma : [a,b] \to G$ be a continuous function.

γ is a path in G. $\{\gamma\} = \gamma([a,b])$ is support of γ.

γ is smooth, if $\gamma \in C^1([a,b])$ and $\gamma'(t) \neq 0, \forall t \in [a,b]$.

- Let $f \in \mathcal{H}(G)$ be s.t. $f'(z) \neq 0, \forall z \in G$,

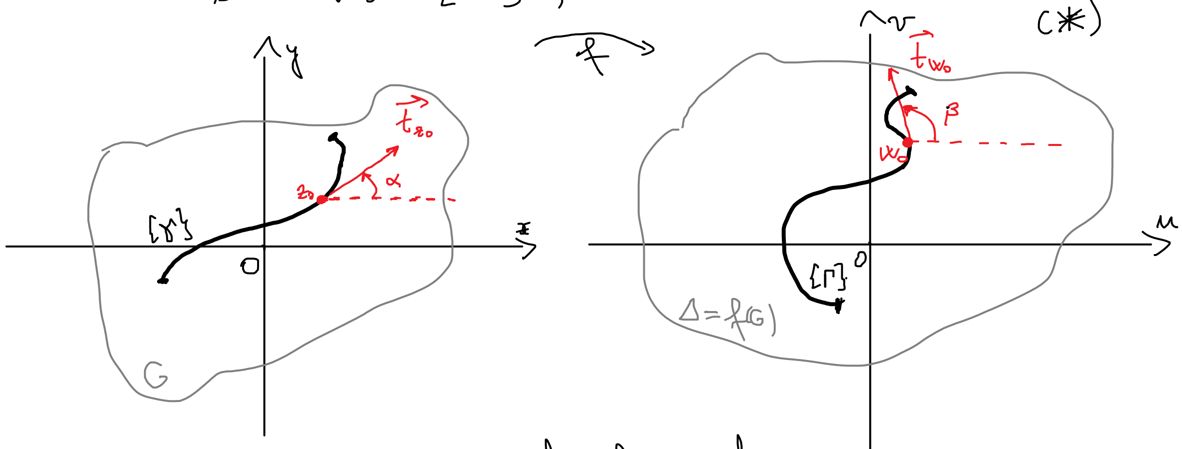
  $\Delta = f(G)$ and fix $z_0 \in G$, $w_0 = f(z_0) \in \Delta$.

- Let γ be a smooth path in G s.t. $z_0 \in \{\gamma\}$

  $\Rightarrow \exists t_0 \in [a,b]$ s.t. $\gamma(t_0) = z_0$.

- Let $\Gamma = f \circ \gamma \Rightarrow \Gamma$ is a smooth path in Δ

  s.t. $w_0 \in \{\Gamma\}$, $w_0 = f(\gamma(t_0))$, and $\Gamma'(t_0) = f'(z_0) \cdot \gamma'(t_0)$. $(*)$



- $\vec{t}_{z_0} = \gamma'(t_0)$ is the tangent of γ at z_0

  $\vec{t}_{w_0} = \Gamma'(t_0)$ is the tangent of Γ at w_0

$\boxed{R_1}$ $(*) \Rightarrow |\Gamma'(t_0)| = |f'(z_0)| \cdot |\gamma'(t_0)|$.

$k_{z_0} = |f'(z_0)| \Rightarrow |\vec{t}_{w_0}| = k_{z_0} \cdot |\vec{t}_{z_0}|$

↳ is the coefficient of linear deformation at z_0 and it is independent of γ.

So, f contracts / expands by the factor k_{z_0} the tangent vectors at z_0, if $k_{z_0} \neq 1$.

- If $k_{z_0} < 1$, then we have a contraction at z_0.

$\{z \in G : |f'(z)| < 1\}$ is the set that is contracted by f.

- If $k_{z_0} > 1$, then we have an expansion at z_0.
  $\{z \in G : |f'(z)| > 1\}$ is the set that is expanded by f.

$\boxed{R2}$ Let $\theta_{z_0} = \arg f'(z_0) \overset{(*)}{\Longrightarrow} \operatorname{Arg} \Gamma'(t_0) = \theta_{z_0} + \operatorname{Arg} \gamma'(t_0)$

$$\Longrightarrow \underbrace{\arg \vec{t}_{w_0}}_{\beta} = \theta_{z_0} + \underbrace{\arg \vec{t}_{z_0}}_{\alpha} \pmod{2\pi}.$$

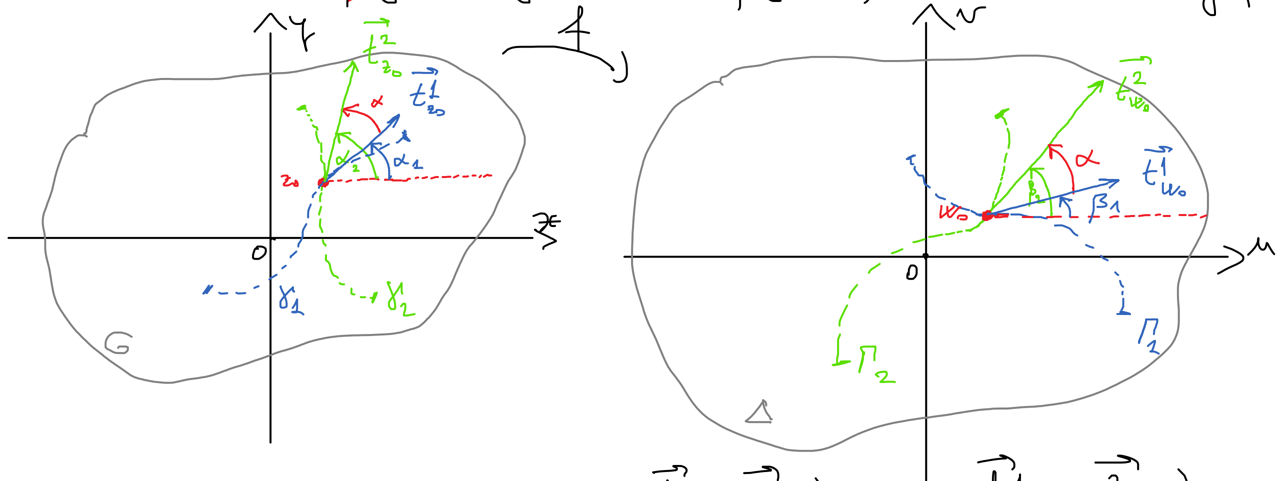
So, θ_{z_0} is the angle of rotation of $\vec{t}_{z_0}$ through f.

Next, let γ_1, γ_2 be smooth paths in G,

$\gamma_1(t_0) = \gamma_2(t_0) = z_0$, $\Gamma_j = f \circ \gamma_j$, $j = 1, 2$.

Let $\alpha_j \in \operatorname{Arg} \gamma_j'(t_0)$, $\beta_j \in \operatorname{Arg} \Gamma_j'(t_0)$ be s.t.

$$\beta_j = \alpha_j + \theta_{z_0}, \quad j = 1, 2 \quad (\text{where } \theta_{z_0} = \arg f'(z_0)).$$

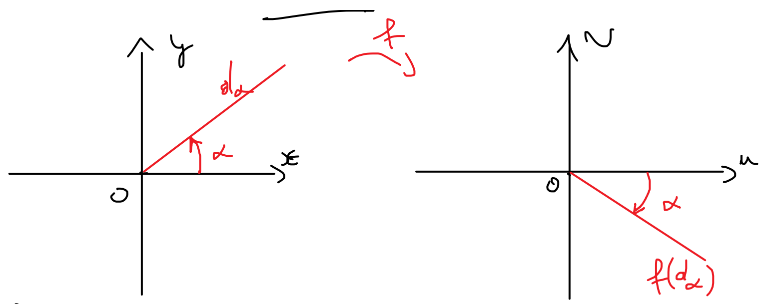


$$\beta_2 - \beta_1 = \alpha_2 - \alpha_1 \Longrightarrow \measuredangle(\underset{\gamma_1'(t_0)}{\underset{\parallel}{\vec{t}^{\,1}_{z_0}}}, \underset{\gamma_2'(t_0)}{\underset{\parallel}{\vec{t}^{\,2}_{z_0}}}) = \measuredangle(\underset{\Gamma_1'(t_0)}{\underset{\parallel}{\vec{t}^{\,1}_{w_0}}}, \underset{\Gamma_2'(t_0)}{\underset{\parallel}{\vec{t}^{\,2}_{w_0}}}) \pmod{2\pi}$$

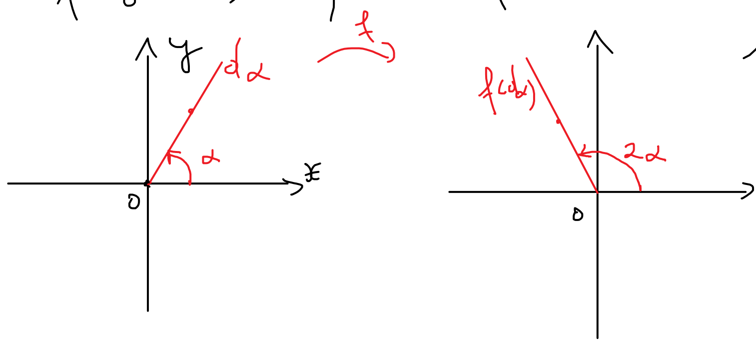
So, $f \in \mathcal{H}(G)$ with $f'(z) \neq 0, \forall z \in G$, preserves the measures and the orientation of the angles between smooth paths. Such a map is called conformal.

$\boxed{Ex1}$ $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$, $z \in \mathbb{C}$.

f preserves the measures of angles, but not the orientation.



$\boxed{Ex_2}$ $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$, $z \in \mathbb{C}$.
$f \in \mathcal{H}(\mathbb{C})$, but $f'(0) = 0$.



f doubles the angles at 0.

Linear fractional / Möbius transformations

$\boxed{D_2}$ Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, $f(z) = \dfrac{az+b}{cz+d}$, $z \in \mathbb{C}_\infty$,
where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$.
f is a linear fractional or Möbius transformation.

$\boxed{R_1}$ • $\boxed{c = 0}$ $\Rightarrow$ $f(z) = \frac{a}{d} z + \frac{b}{d}$, $z \in \mathbb{C}_\infty$. So, $\lim\limits_{z \to \infty} f(z) = f(\infty) = \infty$.
Also, $f \in \mathcal{H}(\mathbb{C})$ and $f'(z) = \frac{a}{d}$, $z \in \mathbb{C}$. Since $\frac{a}{d} \neq 0$,
f is conformal.

• $\boxed{c \neq 0}$ $\Rightarrow$ $\lim\limits_{z \to \infty} f(z) = f(\infty) = \frac{a}{c}$, $f \in \mathcal{H}(\mathbb{C} \setminus \{-\frac{d}{c}\})$,
$$f'(z) = \frac{ad - bc}{(cz+d)^2} \neq 0, \quad \forall z \in \mathbb{C} \setminus \left\{-\frac{d}{c}\right\},$$
so f is conformal.

$z_0 = -\frac{d}{c}$ is called the pole of f ($f(z_0) = \infty$).

$\boxed{P_1}$ The Möbius transform. are homeomorphisms (cont. + bijective + cont. inverse) from $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$ that map the generalized circles to generalized circles.

$f(z) = \dfrac{az+b}{\ }$, $z \in \mathbb{C}_\infty$.

generalized circles to ge

Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, $f(z) = \dfrac{az+b}{cz+d}$, $z \in \mathbb{C}_\infty$.

- $c = 0$: circle $\xrightarrow{f}$ circle and line $\xrightarrow{f}$ line
- $c \neq 0$: $\underbrace{z_0 = -\dfrac{d}{c}}_{\text{the pole of } f}$
  - $\in$ circle $\xrightarrow{f}$ line
  - $\notin$ circle $\xrightarrow{f}$ circle
  - $\in$ line $\xrightarrow{f}$ line
  - $\notin$ line $\xrightarrow{f}$ circle.

$\boxed{R_2}$ If H is the family of all Möbius transformations, then $(H, \circ)$ is group ($\circ$ is the composition btw two functions).

$\boxed{D_3}$ Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct.

$\underline{(z_1, z_2, z_3, z_4)} = \underline{\dfrac{z_1 - z_2}{z_1 - z_4} : \dfrac{z_3 - z_2}{z_3 - z_4}}$ is the <u>cross-ratio</u> of z_1, z_2, z_3, z_4.

$\boxed{P_2}$ Möbius transform. <u>preserve</u> the <u>cross-ratio</u> of any 4 distinct points in $\mathbb{C}_\infty$.

$f \in H$ and $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ distinct

$\Rightarrow (z_1, z_2, z_3, z_4) = (f(z_1), f(z_2), f(z_3), f(z_4))$.

$\boxed{R_3}$ If $z_1, z_2, z_3 \in \mathbb{C}_\infty$ are distinct and $w_1, w_2, w_3 \in \mathbb{C}_\infty$ are distinct

then $\exists!\, f \in H$ s.t. $f(z_j) = w_j$, $j \in \{1, 2, 3\}$.

⚠ f is obtained by solving the eq.:

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3)$$

$\Rightarrow w = f(z)$ is the sol.