Course 9

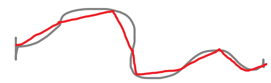
Monday, December 6, 2021 6:30 PM

# Paths

$\boxed{D1}$ Let $[a,b] \subseteq \mathbb{R}$.

$\Delta_n = (t_0 < t_1 < \dots < t_{n+1})$ is a division of $[a,b]$, where $t_0 = a$ and $t_{n+1} = b$.

$Div[a,b] = \{\Delta_n : \Delta_n \text{ is a division of } [a,b],\ n \in \mathbb{N}\}$.



Let $\gamma : [a,b] \to \mathbb{C}$.

$$V(\gamma; \Delta_n) = \sum_{k=1}^{n+1} |\gamma(t_k) - \gamma(t_{k-1})|$$ is the variation of $\gamma$ w.r.t. $\Delta_n \in Div[a,b]$.

$$V(\gamma) = \sup_{\Delta_n \in Div[a,b]} V(\gamma; \Delta_n)$$ is the total variation of $\gamma$.

$\gamma$ is of bounded variation, if $V(\gamma) < \infty$ (is finite).

$\boxed{D2}$ Let $G \subseteq \mathbb{C}$. $\gamma : [a,b] \to \mathbb{C}$ is a piecewise continuously differentiable path in $G$ ($\gamma \in C^1_p([a,b], G)$) if $\{\gamma\} = \gamma([a,b]) \subseteq G$, $\gamma$ is continuous on $[a,b]$ and $\exists\, \Delta_n = (t_0 < t_1 < \dots < t_{n+1}) \in Div[a,b]$, where $t_0 = a$ and $t_{n+1} = b$, s.t.

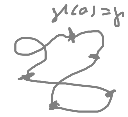
$\gamma|_{[t_k, t_{k+1}]}$ is $C^1$ on $[t_k, t_{k+1}]$, $k = \overline{0, n}$.

(e.g.  polygon)

$\boxed{R1}$ If $\gamma \in C^1_p([a,b], \mathbb{C})$, then $\gamma$ is of bounded variation (or rectifiable) and $l(\gamma) = V(\gamma)$ is the (arc) length of $\gamma$.

$$\left(! \quad l(\gamma) = \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} |\gamma'(t)|\, dt\right)$$

$\gamma(a) = \gamma(b)$



$\gamma(a) = \gamma(b)$

$\boxed{D3}$ $\gamma \in C^1_p([a,b], \mathbb{C})$ is a contour, if $\gamma(a) = \gamma(b)$.

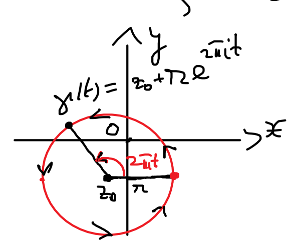
If, in addition, $\gamma|_{[a,b)}$ is injective, then $\gamma$ is a Jordan contour.

$\boxed{Ex.1}$ Let $z_0 \in \mathbb{C}$, $r > 0$, $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = z_0 + r \cdot e^{2\pi i t}$, $t \in [0,1]$

Then $\gamma$ is a Jordan contour with

$$\{\gamma\} = \partial U(z_0, r).$$

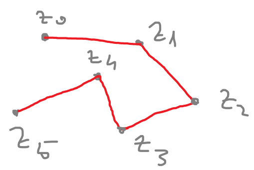
$$\gamma'(t) = r \cdot e^{2\pi i t} \cdot 2\pi i = 2\pi i r\, e^{2\pi i t},\quad t \in [0,1].$$



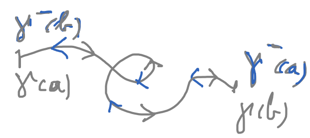

$\boxed{!}$ If $G \subseteq \mathbb{C}$ open, $f \in \mathcal{H}(G)$, $\alpha \in C^1([a,b], G)$, then $(f \circ \alpha)'(t) = f'(\alpha(t)) \cdot \alpha'(t)$, $t \in [a,b]$, so $f \circ \alpha \in C^1([a,b], f(G))$.

$\boxed{D_4}$ Let $\Delta_n = (\underset{a}{\underset{\shortparallel}{t_0}} < t_1 < \ldots < \underset{b}{\underset{\shortparallel}{t_{n+1}}}) \in Div[a,b]$ and
$\gamma_k \in C_p^1([t_k, t_{k+1}], \mathbb{C})$, $k = \overline{0,n}$, be s.t.
$\gamma_k(t_{k+1}) = \gamma_{k+1}(t_{k+1})$, $k = \overline{0, n-1}$.
$\gamma \overset{not}{=} \gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_n$, called the union path of $\gamma_0, \ldots, \gamma_n$,
is such that $\gamma(t) = \gamma_k(t)$, $t \in [t_k, t_{k+1}]$, $k = \overline{0,n}$.
(!) $\gamma = \gamma_0 \cup \ldots \cup \gamma_n \in C_p^1([a,b], \mathbb{C})$.

$\boxed{Ex.2}$ Let $z_0, z_1, \ldots, z_{n+1} \in \mathbb{C}$, $\Delta_n = (t_0 < t_1 < \ldots < t_{n+1}) \in Div[a,b]$
and $\lambda_k(t) = \dfrac{t_{k+1} - t}{t_{k+1} - t_k} \cdot z_k + \dfrac{t - t_k}{t_{k+1} - t_k} \cdot z_{k+1}$, $t \in [t_k, t_{k+1}]$, $k = \overline{0,n}$.
Then $\lambda = \lambda_0 \cup \ldots \cup \lambda_n \in C_p^1([a,b], \mathbb{C})$ and
$\{\lambda\} = [z_0, z_1] \cup [z_1, z_2] \cup \ldots \cup [z_n, z_{n+1}]$ is a polygon.




$\boxed{D_5}$ Let $\gamma \in C_p^1([a,b], \mathbb{C})$.
$\gamma^-$, called the opposite path of $\gamma$, is such that
$\gamma^-(t) = \gamma(a+b-t)$, $t \in [a,b]$.




$\boxed{D_6}$ Let $\gamma_1 \in C_p^1([a_1, b_1], \mathbb{C})$.
$\gamma_2 \in C_p^1([a_2, b_2], \mathbb{C})$ is equivalent with $\gamma_1$, if
$\exists\, h : [a_1, b_1] \to [a_2, b_2]$ is an increasing homeomorphism
s.t. $\gamma_1 = \gamma_2 \circ h$. (!) $\{\gamma_1\} = \{\gamma_2\}$.

## Complex integral

$\boxed{D_7}$ Let $\gamma \in C_p^1([a,b], \mathbb{C})$, $f : \{\gamma\} \to \mathbb{C}$ be continuous.
The complex integral (Cauchy's integral) of $f$ along $\gamma$ is
$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,d\gamma(t).$$

along $\gamma$ is

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,d\gamma(t).$$

$\boxed{R_2}$ If $\gamma \in C^1_p([a,b],\mathbb{C})$ has the div. $\Delta_n = (t_0 < t_1 < \dots < t_{n+1}) \in \mathcal{D}iv[a,b]$, then

$$\int_a^b f(\gamma(t))\,d\gamma(t) = \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f(\gamma(t))\cdot\gamma'(t)\,dt.$$

$\boxed{Ex.3}$ Let $z_0 \in \mathbb{C}$, $r>0$, $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = z_0 + r e^{2\pi i t}$, $t \in [0,1]$,
$f : \mathbb{C}\setminus\{z_0\} \to \mathbb{C}$, $f(z) = \frac{1}{z-z_0}$, $z \in \mathbb{C}\setminus\{z_0\}$.

$$\int_\gamma f = \int_\gamma \frac{1}{z-z_0}\,dz = \int_0^1 \frac{1}{\gamma(t)-z_0}\cdot\gamma'(t)\,dt \underset{\boxed{Ex.1}}{=} \int_0^1 \frac{1}{\cancel{r e^{2\pi i t}}}\cdot 2\pi i\cancel{r}\cdot\cancel{e^{2\pi i t}}\,dt =$$

$$= \int_0^1 2\pi i\,dt = 2\pi i \cdot t\Big|_0^1 = 2\pi i.$$

$\boxed{P_1}$
- $\int_\gamma(\alpha f + \beta g) = \alpha\int_\gamma f + \beta\int_\gamma g$, $\alpha,\beta \in \mathbb{C}$.
- $\int_{\gamma^-} f = -\int_\gamma f$
- $\int_{\gamma_0\cup\dots\cup\gamma_n} f = \int_{\gamma_0} f + \dots + \int_{\gamma_n} f$
- $\gamma_2$ is equivalent with $\gamma_1$ $\Rightarrow$ $\int_{\gamma_1} f = \int_{\gamma_2} f$.
- $\left|\int_\gamma f\right| \le M\cdot \ell(\gamma)$, where $M = \max_{z\in\{\gamma\}} |f(z)|$.
- if $\underset{\hookrightarrow \text{cont.}}{f_n} : \{\gamma\} \to \mathbb{C}$, $n \in \mathbb{N}$, converges uniformly on $\{\gamma\}$ to $f$, then $\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f$.

$\boxed{D_8}$ Let $G \subseteq \mathbb{C}$ be open and $f : G \to \mathbb{C}$.
$g : G \to \mathbb{C}$ is a primitive of $f$ on $G$, if
$g \in \mathcal{H}(G)$ and $g' = f$.

$\boxed{T_1}$ (the connection between the primitive and the integral)
Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be continuous.
$\Rightarrow$ If $\int_\gamma f = 0$, $\forall$ contour in $D$, then $f$ has a primitive on $D$.

Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$

i) If $\int_\gamma f = 0$, $\forall \gamma$ contour in $D$, then $f$ has a primitive on $D$.

ii) If $f$ has a primitive $g$ on $D$, then $\int_\gamma f = g(\gamma(b)) - g(\gamma(a))$, $\forall \gamma \in C^1_p([a,b], D)$.

$\boxed{C_1}$ If $D \subseteq \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is cont., then:
$f$ has a primitive on $D \iff \int_\gamma f = 0$, $\forall \gamma$ contour in $D$.