COURSE 1

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final grade = 40% seminar $\begin{cases} 10\% \text{ seminar answers} \\ 30\% \text{ midterm exam} \end{cases}$

60% final exam

COMPLEX NUMBERS

Consider the following operations on $\mathbb{R}^2$:

$$\boxed{\begin{aligned} &(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ &(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \end{aligned}}$$

It can be proved that $(\mathbb{R}^2, +, \cdot)$ is a comm. field which we denote by $\mathbb{C}$.

$(\mathbb{R} \times \{0\}, +, \cdot)$ is a subfield of $\mathbb{C}$ and $\varphi : \mathbb{R} \to \mathbb{R} \times \{0\}$ given by $\varphi(x) = (x, 0)$

φ is an isomorphism w.r.t. the usual addition and multiplication on $\mathbb{R}$

- We identify $\mathbb{R}$ with $\mathbb{R} \times \{0\}$, so $\boxed{\mathbb{R} \subseteq \mathbb{C}}$ subfield
- Denote $i = (0,1)$ called the imaginary unit

$\forall (x,y) \in \mathbb{C} : (x,y) = (x,0) + (0,y)$

$= x + (y,0) \cdot (0,1)$ $\qquad x = (x,0)$

$$\boxed{(x,y) = x + y \cdot i}$$

$\forall z \in \mathbb{C} : \exists! \, x \in \mathbb{R}, \exists! \, y \in \mathbb{R}$ s.t. $\boxed{z = x + iy}$

Remark: $i^2 = i \cdot i = (0,1) \cdot (0,1) = (0 - 1, 0 + 0) = (-1, 0) = -1$

$$\boxed{i^2 = -1}$$

Remark: $\mathbb{C}$ is not an ordered field!

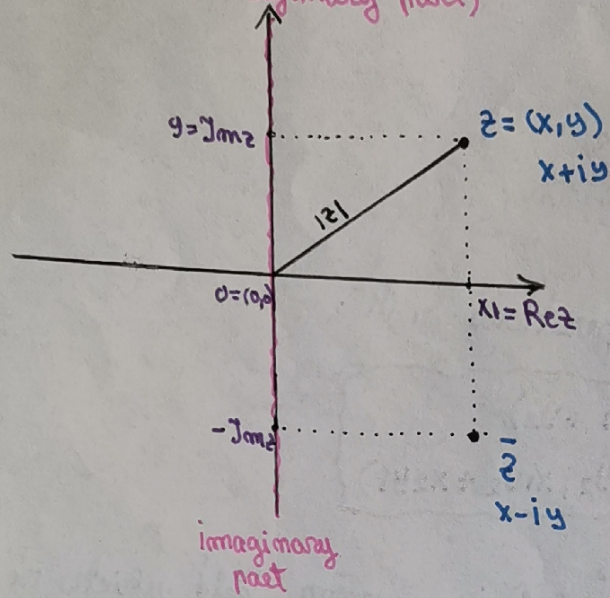
We cannot compare complex numbers!

THE ALGEBRAIC FORM

• Let $z = x + iy \in \mathbb{C}$

$\operatorname{Re} z = x$ real part

$\operatorname{Im} z = y$ (imaginary part)



$\boxed{|z| = \sqrt{x^2 + y^2}}$ the distance

modulus - abs. value

$\boxed{\bar{z} = x - iy}$ conjugate of z

$\boxed{\operatorname{Re} z = \frac{z + \bar{z}}{2}} \in \mathbb{R}$

$\boxed{\operatorname{Im} z = \frac{z - \bar{z}}{2i}} \in \mathbb{R}$

$\boxed{|z| = |\bar{z}|}$

$\boxed{|z_1 \cdot z_2| = |z_1| \cdot |z_2|}$ $\forall z_1, z_2 \in \mathbb{C}$

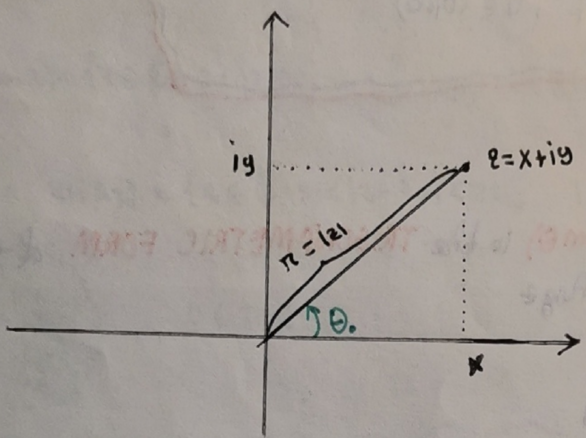
$\boxed{|z|^2 = z \cdot \bar{z}}$

$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$

~ THE TRIGONOMETRIC FORM ~

Let $z \in \mathbb{C}$ (where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$
$= \mathbb{C} \setminus \{(0,0)\}$)

• Denote $|z| = r > 0$



(r, θ) are polar coordinates

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

$\theta_0 = \mathrm{Arg}\, z$ = the angle between $\vec{O_z}$ and the position real semiaxis

Every $\theta \in \mathbb{R}$ that satisfies

(*) $z = r(\cos\theta + i\sin\theta)$ is called an ARGUMENT of Z

• Remark: $z = 0$ has no argument

$\exists!\, \theta_0 \in (-\pi, \pi]$ s.t. (*) holds which is called the PRINCIPAL ARGUMENT and we denote $\boxed{\theta_0 = \arg z}$

$\mathrm{Arg}\, z := \{\arg z + 2k\pi : k \in \mathbb{Z}\}$ is the set of all arguments of z

$\mathrm{Arg}\ \mathbb{C}^* \to \mathcal{P}(\mathbb{R})$ is called the MULTIVALUE ARGUMENT FUNCTION

The set of all subsets of $\mathbb{R}$

PROPERTIES:

$$\boxed{\begin{aligned}&\operatorname{Arg}(z_1\cdot z_2)=\operatorname{Arg}z_1+\operatorname{Arg}z_2\\ &\quad(\Leftrightarrow)\ \arg(z_1\cdot z_2)=\arg z_1+\arg z_2\ (\bmod 2\pi))\\ &\operatorname{Arg}\left(\frac{z_1}{z_2}\right)=\operatorname{Arg}z_1-\operatorname{Arg}z_2,\ \forall z_1,z_2\in\mathbb{C}^*\\ &\operatorname{Arg}\frac{1}{z}=-\operatorname{Arg}z=\operatorname{Arg}\bar{z},\ z\in\mathbb{C}^*\\ &\operatorname{Arg}(r\cdot z)=\operatorname{Arg}z,\ z\in\mathbb{C}^*,\ r\in(0,\infty)\end{aligned}}$$

<u>Remark</u>:

$\forall z\in\mathbb{C}^*$: $z=r(\cos\theta+i\sin\theta)$ is the TRIGONOMETRIC FORM of z
$r=|z|$ $\theta\in\operatorname{Arg}z$

Ex: $z_1=r_1(\cos\theta_1+i\sin\theta_1)$
$z_2=r_2(\cos\theta_2+i\sin\theta_2)$

$$\boxed{z_1\cdot z_2=r_1\cdot r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))}$$

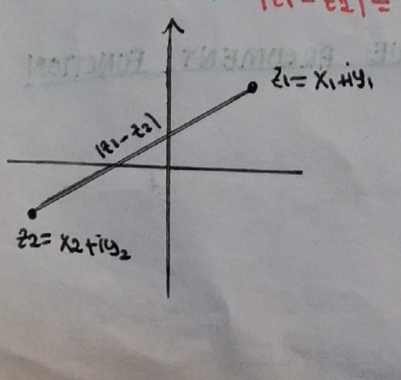
multiplying of 2 complex numbers in trigonometric form

– TOPOLOGY OF THE COMPLEX PLANE –

$\forall P(x,y)\in\mathbb{R}^2$ is identified with the complex numbers $\boxed{z=x+iy}$ called AFFIX of P

- The complex plane is the set of complex numbers with the Euclidean structure of $\mathbb{R}^2$ which we denote again by $\mathbb{C}$.

- If $z_1=x_1+iy_1$, $z_2=x_2+iy_2\in\mathbb{C}$, then the EUCLIDEAN DISTANCE between z_1 and z_2 is $|z_1-z_2|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$



CURS I

ANALIZĂ

• For $z_0 \in \mathbb{C}, r>0$ $\quad r \in (0,\infty)$

$U(z_0,r) = \{z \in \mathbb{C} : |z-z_0| < r\}$ open disk

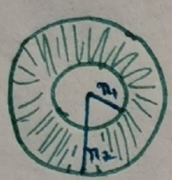
$\bar{U}(z_0,r) = \{z \in \mathbb{C} : |z-z_0| \le r\}$ CLOSED DISK

$\partial U(z_0,r) = \{z \in \mathbb{C} : |z-z_0| = r\}$ CIRCLE

(a disk without a center)

$\dot{U}(z_0,r) = \{z \in \mathbb{C} : 0 < |z-z_0| < r\}$ PUNCTURED DISK

$U(z_0; r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z-z_0| < r_2\}$ THE ANNULUS



$0 \le r_1 < r_2$

Remark: $\dot{U}(z_0,r) = U(z_0; 0, r)$

• Let $A \subseteq \mathbb{C}, z_0 \in \mathbb{C}$

• Recall from Calculus in $\mathbb{R}^n$

z_0 is an: • INTERIOR POINT of A if $\exists r>0$ s.t. $U(z_0,r) \subseteq A$

• CLOSURE POINT of A if: $\forall r>0 : U(z_0,r) \cap A \neq \emptyset$

• ACCUMULATION POINT of A if $\forall r>0 : \dot{U}(z_0,r) \cap A \neq \emptyset$

• BOUNDARY POINT of A if $\forall r>0$ $U(z_0,r) \cap A \neq \emptyset$

$U(z_0,r) \cap (\mathbb{C} \setminus A) \neq \emptyset$

A is OPEN if $\forall z \in A$ is an INTERIOR point of A

CLOSED if $\forall z \in A$ is a CLOSURE point of A

BOUNDED if $\exists r>0$ s.t. $A \subseteq U(z_0,r)$