CURS 2
ANALIZĂ

TOPOLOGY OF THE COMPLEX PLANE

$A \subseteq \mathbb{C}$ is:

- **OPEN** if $\forall z \in A$ is an interior point $\Leftrightarrow A = \text{int } A$
- **CLOSED** if every closure point of A is in $A \Leftrightarrow A = \text{cl } A = \overline{A}$

NOTATIONS: A' = is the set of ACCUMULATION POINTS of A

∂A = THE BOUNDARY OF A

PROPERTIES: $A = \bar{A} \Leftrightarrow \mathbb{C} \setminus A$ is open (a set is closed if the complementary is open)

$\bar{A} = A \cup A'$

$\partial A = \bar{A} \setminus \text{int } A$

! $A \subseteq \mathbb{C}$ is COMPACT $\Leftrightarrow$ A is CLOSED and BOUNDED

Def: A SEQUENCE $(z_n)_{n \in \mathbb{N}^*}$ in $\mathbb{C}$ is CONVERGENT to a point z_0 in $\mathbb{C}$ ($z_0 \in \mathbb{C}$), if

$$\lim_{n \to \infty} |z_n - z_0| = 0$$

NOTATION: $z_n \to z_0$ or $\lim_{n \to \infty} z_n = z_0$

Remark: If $z_n = x_n + i y_n$, $n \in \mathbb{N}^*$ and $z_0 = x_0 + i y_0$, then:

$$\lim_{n \to \infty} z_n = z_0 \Leftrightarrow \lim_{n \to \infty} x_n = x_0 \text{ and } \lim_{n \to \infty} y_n = y_0$$

PROPOSITION : Let $A \subseteq \mathbb{C}$, $A \neq \emptyset$

Then:

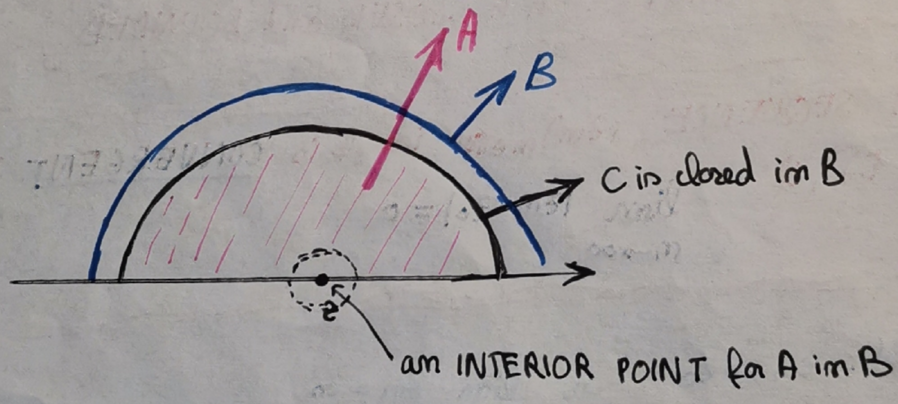
- $z_0 \in \bar{A} \iff \exists (z_n)_{n\in\mathbb{N}^*}$ in A s.t. $\boxed{\lim_{n\to\infty} z_n = z_0}$ ← CLOSURE POINT
- $z_0 \in A' \iff \exists (z_n)_{n\in\mathbb{N}^*}$ in $A\setminus\{z_0\}$ s.t. $\boxed{\lim_{n\to\infty} z_n = z_0}$ ← LIMIT POINT or ACCUMULATION POINT
- A is COMPACT $\iff \forall (z_n)_{n\in\mathbb{N}^*}$ in A, $\exists (z_{n_k})_{k\in\mathbb{N}^*}$ in A and $\exists z_0 \in A$ s.t. $\boxed{\lim_{k\to\infty} z_{n_k} = z_0}$

  ↑ convergent subsequence

THE INDUCED TOPOLOGY

Let $A \subseteq B \subseteq \mathbb{C}$

Def A is OPEN in B if $\forall z \in A\ \exists r>0$ s.t. $U(z,r) \cap B \subseteq A$



B is NOT OPEN in $\mathbb{C}$, but A is open B

A is CLOSED in B if $B\setminus A$ is OPEN in B

REMARK A is CLOSED in $B \iff$

$\forall (z_n)_{n\in\mathbb{N}^*}$ in A with $\lim_{n\to\infty} z_n = z_0 \in B$, we have $z_0 \in A$

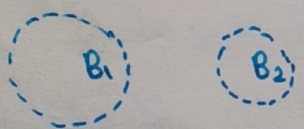
CONNECTED SETS IN $\mathbb{C}$

• Let $B \subseteq \mathbb{C}$, $B \neq \emptyset$

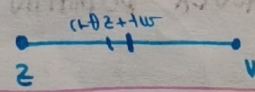
DEF B is CONNECTED, if the following holds:
if $A \subseteq B$ and A is opend and CLOSED in B, then either $A = B$ or $A = \emptyset$

REMARK: B is NOT CONNECTED $\iff$
$\exists B_1, B_2$ in B which are not empty and open in B st
$$\boxed{\begin{array}{l} B_1 \cap B_2 \neq \emptyset \\ B = B_1 \cup B_2 \end{array}}$$

EXAMPLE: if B is the UNION of 2 disjoint open disks then B is NOT CONNECTED

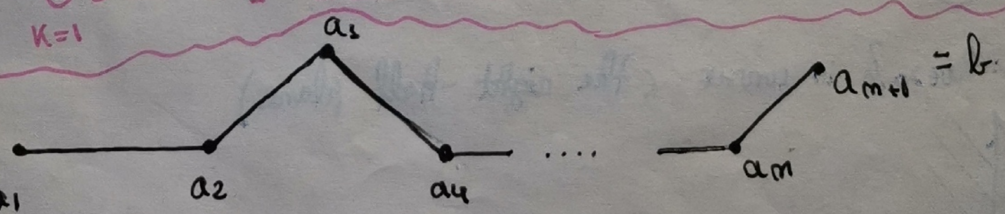


NOTATION for $z, w \in \mathbb{C}$ $[z,w] = \{(1-t)z + tw : t \in [0,1]\}$ is the

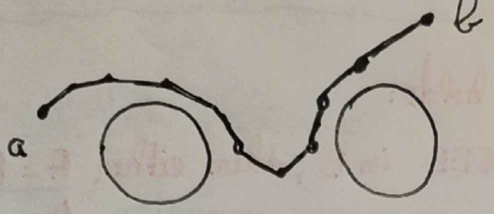


SEGMENT between z and w

Def A POLYGON from $a \in \mathbb{C}$ to $b \in \mathbb{C}$ is a set of the form
$\bigcup_{k=1}^{m} [a_k, a_{k+1}]$ where $a_1 = a$



Def: $A \subseteq \mathbb{C}$ is POLYGONALLY CONNECTED if $\forall a,b \in A$ $\exists$ a polygon in A from a to b

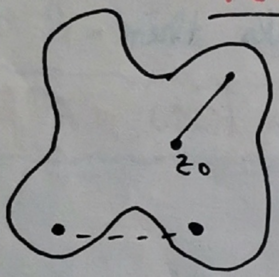


PROPOSITION Let $D \subseteq \mathbb{C}$ be open. Then D is connected $\Longleftrightarrow$ D is polygonally connected

Def $D \subseteq C$ is a DOMAIN if D is OPEN and CONNECTED

$D \subseteq C$ is a STARLIKE DOMAIN w.r.t a point $z_0 \in D$ if $\forall z \in D: [z_0, z] \subset D$

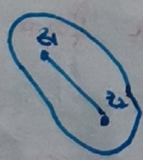
fixed point



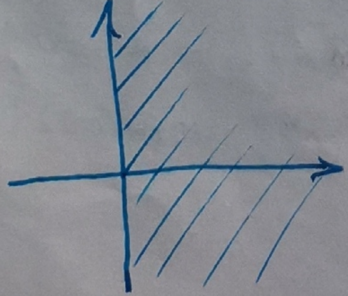
starlike but not convex

$D \subseteq \mathbb{C}$ is a CONVEX DOMAIN if $\forall z_1, z_2 \in D: [z_1, z_2] \subset D$

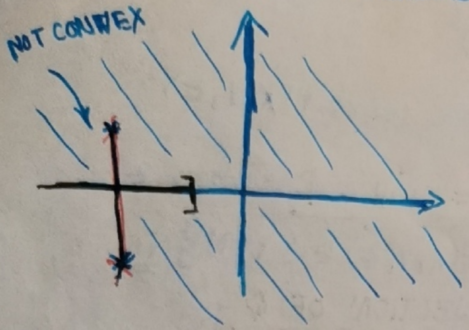
Examples: • $U(z_0, r)$ is convex in $\mathbb{C}$ $\forall z_0 \in \mathbb{C}, r > 0$



• $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is convex (the right-half plane)



- $\mathbb{C} \setminus (-\infty, -1]$ is STARLIKE w.r.t. to 0 but it is not CONVEX



THE STEREOGRAPHIC PROJECTION

We shall consider the ONE POINT COMPACTIFICATION of $\mathbb{C}$ by adjoining a point, denoted by ∞, which is NOT in $\mathbb{C}$ such that $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is a compact SPACE

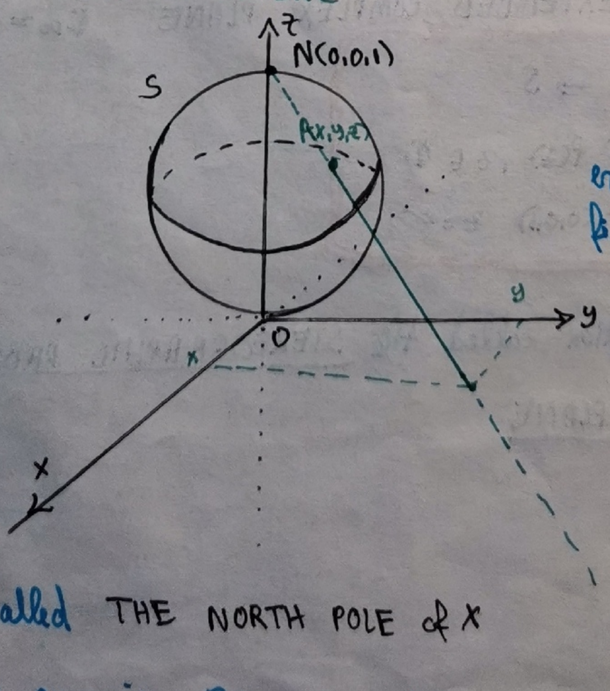
EXTENDED COMPLEX PLANE

Let $S \subseteq \mathbb{R}^3$ be the SPHERE $(S): (x-0)^2 + (y-0)^2 + (z-\frac{1}{2})^2 = \frac{1}{4}$ and identify the COMPLEX PLANE with the plane XOY.

for 3D coord

S is called THE RIENMAN SPHERE

In $\mathbb{R}^3$



we can do this for every point in the complex field

$N(0,0,1)$ is called THE NORTH POLE of X

- fiecărui punct $z = x+iy \in \mathbb{C}$ îi corespunde în mod unic un punct $P(x,y,z) \in S \setminus \{N\}$

There is a one-to-one correspondence between the points in $\mathbb{C}$ and the points in $S\setminus\{N\}$

- if $z = x+iy \in \mathbb{C}$ is the affix of the point $M(x,y,0)$ the line NM intersects $S\setminus\{N\}$ ~~exactly~~ at one point $P(X,Y,Z)$ so $\varphi : \mathbb{C} \to S\setminus\{N\}$ given by $\varphi(z) = P(X,Y,Z)$ $z = x+iy \in \mathbb{C}$ where z is the affix of the stereographic projection of $P(X,Y,Z) \in S\setminus\{N\}$ is bijective and is called THE STEREOGRAPHIC PROJECTION OF $\mathbb{C}$

PROPOSITION : $\varphi : \mathbb{C} \to S\setminus\{N\}$ $\varphi(z) = \left(\underbrace{\frac{\operatorname{Re} z}{1+|z|^2}}_{X}, \underbrace{\frac{\operatorname{Im} z}{1+|z|^2}}_{Y}, \underbrace{\frac{|z|^2}{1+|z|^2}}_{Z} \right) z \in \mathbb{C}$

$$\varphi^{-1}(X,Y,Z) = \frac{X+iY}{1-Z} \quad (X,Y,Z) \in S\setminus\{N\}$$

We consider the stereographic proj. of the north pole to be the point at infinity $N(0,0,1) \xrightarrow{\text{associate}} \infty \notin \mathbb{C}$

abstract point

We obtain the EXTENDED COMPLEX PLANE $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$

Then $\tilde{\varphi} : \mathbb{C}_\infty \to S$

$$\tilde{\varphi}(z) = \begin{cases} \varphi(z), & z \in \mathbb{C} \\ (0,0,1) & z = \infty \end{cases}$$

bijective function called the STEREOGRAPHIC PROJECTION OF THE EXTENDED COMPLEX PLANE