CURS 3
ANALIZĂ

Def $V \subseteq \mathbb{C}_\infty$ is a NEIGHBOURHOOD of ∞ if $\exists r>0$ s.t. $\{z\in\mathbb{C} : |z|>r\}\cup\{\infty\}\subseteq V$.

$V\subseteq \mathbb{C}_\infty$ is a neighbourhood of $z_0\in\mathbb{C}$ if
$\exists r>0$ s.t. $\{z\in\mathbb{C} : |z-z_0|<r\}\subseteq V$

NOTATION:
$\mathcal{V}(z)$ - the family of neighbourhoods of $z\in\mathbb{C}_\infty$

If $P(x,y,z)=\tilde{\varphi}(z)\in S\setminus\{N\}$, $z\in\mathbb{C}$ then the EUCLIDEAN DISTANCE between N and P is

$$\|NP\|=\sqrt{(x-0)^2+(y-0)^2+(z-1)^2}=\sqrt{\underbrace{x^2+y^2+z^2}_{z}-2z+1}=\sqrt{1-z}=$$

$$=\sqrt{1-\frac{|z|^2}{|z|^2+1}}=\frac{1}{\sqrt{|z|^2+1}}$$

So $|z|>r \Longleftrightarrow \|NP\|<\frac{1}{\sqrt{r^2+1}}$

⚠ $V\in\mathcal{V}(\infty) \Longleftrightarrow \tilde{\varphi}(V)$ is a neighbourhood of N on S

⚠ If $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}$, then:

$\lim\limits_{n\to\infty} z_n=\infty \Longleftrightarrow \forall V\in\mathcal{V}(\infty)\ \exists n_V\in\mathbb{N}$ s.t. $z_n\in V$, $n\geq n_V \Longleftrightarrow$

$\Longleftrightarrow \forall r>0\ \exists n_r\in\mathbb{N}$ s.t. $|z_n|>r$, $n\geq n_r$

$\Longleftrightarrow \lim\limits_{n\to\infty}\underbrace{|z_n|}_{\in\mathbb{R}}=\infty \Longleftrightarrow \lim\limits_{n\to\infty}\frac{1}{|z_n|}=0$

$\Longleftrightarrow \lim\limits_{n\to\infty}\frac{1}{z_n}=0$

Def For $z_1,z_2\in\mathbb{C}_\infty$, let $d_c(z_1,z_2)=\|P_1P_2\|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$
where $P_j(x_j,y_j,z_j)=\tilde{\varphi}(z_j)$ $j\in\{1,2\}$

$d_c:\mathbb{C}_\infty\times\mathbb{C}_\infty\to[0,\infty)$ is called the CHORDAL METRIC on $\mathbb{C}_\infty$

$$d_c(z_1,z_2)=\begin{cases}\dfrac{|z_1-z_2|}{\sqrt{|z_1|^2+1}\sqrt{|z_2|^2+1}} & z_1,z_2\in\mathbb{C}\\[2ex] \dfrac{1}{\sqrt{|z|^2+1}} & z_1=z_2\in\mathbb{C}\ \ z_2=\infty \text{ or } z_1=\infty,\ z_2=z\in\mathbb{C}\\[2ex] 0 & z_1=z_2=\infty\end{cases}$$

- $\lim z_m=\infty \Longleftrightarrow d_c(z_m,\infty)\longrightarrow 0$
- $\lim\limits_{m\to\infty} z_m = z\in\mathbb{C} \Longleftrightarrow d_c(z_m,z)\longrightarrow 0$

$(\mathbb{C}_\infty, d_c)$ is a complete metric space

DEF A GENERALISED CIRCLE in $\mathbb{C}_\infty$ is either a (usual) circle or a line.

The stereograph. proj of a circle in S is a generalized circle in $\mathbb{C}_\infty$

- if the circle passes through N, then it's stereographic proj is a line
- if the circle is not passing through N then it's stereographic proj. is a circle

So, the lines are gen. circles passing through ∞.

COMPLEX FUNCTIONS OF A COMPLEX VARIABLE

Let $A\subseteq\mathbb{C}$

DEF $f:A\to\mathbb{C}$ is called a COMPLEX FUNCTION of a COMPLEX VARIABLE.

We denote:

- $\mathrm{Re}\, f = u$ and $\mathrm{Im}\, f = v$ where $u,v: A\to\mathbb{R}$
- $f=u+iv$ $\qquad f(z)=u(z)+iv(z)\qquad z\in A$

DEF Let $z_0 \in A'$ $(\exists (z_n)_{n\in\mathbb{N}} \subset A\setminus\{z_0\}\ z_n \to z_0)$ and $l \in \mathbb{C}$.
f has limit l at z_0 $(\lim_{z\to z_0} f(z) = l)$ if
$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall z \in A$ with $0 < |z - z_0| < \delta: |f(z) - l| < \varepsilon$

⚠ $\lim_{z\to z_0} f(z) = l \iff \begin{cases} \lim_{z\to z_0} \underbrace{\mathrm{Re} f(z)}_{u} = \mathrm{Re}\, l \\ \lim_{z\to z_0} \underbrace{\mathrm{Im} f(z)}_{v} = \mathrm{Im}\, l \end{cases}$

DEF: Let $z_0 \in A'$. f has limit ∞ at z_0 $(\lim_{z\to z_0} f(z) = \infty)$
if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall z \in A$ with $0 < |z - z_0| < \delta: |f(z)| > \varepsilon$

⚠ $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0$

DEF Let $\infty \in A'$ $(\exists (z_n)_{n\in\mathbb{N}} \subset A$ s.t. $z_n \to \infty)$ and $l \in \mathbb{C}$
f has limit l at ∞ $(\lim_{z\to\infty} f(z) = l)$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall z \in A$ with $|z| > \delta: |f(z) - l| < \varepsilon$

⚠ · $\lim_{z\to\infty} f(z) = \lim_{z\to 0} f\left(\frac{1}{z}\right)$

- $\lim_{z\to z_0} f(z) = 0 \iff \lim_{z\to z_0} |f(z)| = 0$

- $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} |f(z)| = \infty$

- $\lim_{z\to z_0} f(z) = l \in \mathbb{C}_\infty \iff \forall (z_n)_{n\in\mathbb{N}} \subseteq A$ with $\lim_{n\to\infty} z_n = z_0: \lim_{n\to\infty} f(z_n) = l$

DEF Let $A \subseteq \mathbb{C}$, $z_0 \in A \cap A'$ and $f: A \to \mathbb{C}$. f is CONTINUOUS on A if f is continuous at any $z_0 \in A$

⚠ f is cont at $z_0 \iff u = \operatorname{Re} f,\ v = \operatorname{Im} f$ are cont at z_0

⚠ The usual operations for cont. real functions hold for continuous complex functions

DIFFERENTIABILITY IN C

Def 1. Let $J \subseteq \mathbb{R}$ be open, $f: J \to \mathbb{C}$, $t_0 \in J$

f is DIFFERENTIABLE at z_0 if $\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0) \in \mathbb{C}$

⚠ • $f = u + iv : J \to \mathbb{C}$ is diff. at $z_0 \iff u, v : J \to \mathbb{R}$ are diff. at z_0

• $f'(z_0) = u'(z_0) + i v'(z_0)$

Let $G \subseteq \mathbb{C}$ be open

$f: G \to \mathbb{C}$, $z_0 \in G$. f is DIFFERENTIABLE at z_0 if

$\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0) \in \mathbb{C}$ called the derivative of f at z_0

DEF 3 $f: G \to \mathbb{C}$ $z_0 \in G$ f is $\mathbb{C}$-differentiable at z_0 if

$\exists \sigma \in \mathbb{C}, \exists w : G \setminus \{z_0\} \to \mathbb{C}$ s.t. $\lim_{z \to z_0} w(z) = 0$,

$$f(z) = f(z_0) + \sigma(z - z_0) + w(z)|z - z_0|, \quad z \in G \setminus \{z_0\}$$

Def. 3 $f: G \to \mathbb{C}$, $z_0 \in G$. f is $\mathbb{C}$-differentiable at z_0, if $\exists \alpha \in \mathbb{C}$, $\exists \omega: G \setminus \{z_0\} \to \mathbb{C}$ s.t. $\lim_{z \to z_0} \omega(z) = 0$,

$$f(z) = f(z_0) + \alpha(z - z_0) + \omega(z)|z - z_0|, \quad z \in G \setminus \{z_0\}.$$

$\boxed{P_1}$ Let $f: G \to \mathbb{C}$, $z_0 \in G$. Then: f is $\mathbb{C}$-diff. at $z_0 \Longleftrightarrow$ f is diff. at z_0.

Proof: f is $\mathbb{C}$-diff. at $z_0 \Longleftrightarrow \exists \alpha \in \mathbb{C}$ s.t.

$$\lim_{z \to z_0} \underbrace{\left| \underbrace{\frac{f(z) - f(z_0) - \alpha(z - z_0)}{|z - z_0|}}_{\omega(z)} \right|}_{\frac{|f(z) - f(z_0) - \alpha(z - z_0)|}{|z - z_0|} = \left| \frac{f(z) - f(z_0)}{z - z_0} - \alpha \right|} = 0$$

$$\left(\left| \frac{u}{|w|} \right| = \frac{|u|}{|w|} = \left| \frac{u}{w} \right|, \ \forall u, w \in \mathbb{C}, \ w \neq 0 \right)$$

$\Longleftrightarrow \exists \alpha \in \mathbb{C}$ s.t. $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \alpha \Longleftrightarrow f$ is diff. at z_0.

⚠ f is $\mathbb{C}$-diff. at $z_0 \Longrightarrow \alpha = f'(z_0)$. (1)

Def. 4 $f = u + iv: G \to \mathbb{C}$, $z_0 = x_0 + iy_0 \in G$.
f is $\mathbb{R}$-differentiable at z_0, if u, v are Fréchet differentiable at (x_0, y_0).

Recall: • $\|(x,y)\| = \sqrt{x^2+y^2}$, $(x,y)\in\mathbb{C}$ (the Euclidean norm in $\mathbb{R}^2$).

• f is $\mathbb{R}$-diff. at $z_0=(x_0,y_0)$ $\Longleftrightarrow$

$$\Longleftrightarrow \begin{cases} \exists\, a_1,b_1\in\mathbb{R},\ \exists\, \omega_1: G\setminus\{(x_0,y_0)\}\to\mathbb{R} \text{ s.t. } \lim\limits_{(x,y)\to(x_0,y_0)} \omega_1(x,y)=0, \\ u(x,y)=u(x_0,y_0)+a_1(x-x_0)+b_1(y-y_0)+\omega_1(x,y)\|(x-x_0,y-y_0)\|,\ (x,y)\in G\setminus\{(x_0,y_0)\} \\ \text{and} \\ \exists\, a_2,b_2\in\mathbb{R},\ \exists\, \omega_2: G\setminus\{(x_0,y_0)\}\to\mathbb{R} \text{ s.t. } \lim\limits_{(x,y)\to(x_0,y_0)} \omega_2(x,y)=0, \\ v(x,y)=v(x_0,y_0)+a_2(x-x_0)+b_2(y-y_0)+\omega_2(x,y)\|(x-x_0,y-y_0)\|,\ (x,y)\in G\setminus\{(x_0,y_0)\}. \end{cases}$$

Remark 1: • $a_1=\frac{\partial u}{\partial x}(x_0,y_0)$, $a_2=\frac{\partial v}{\partial x}(x_0,y_0)$, $b_1=\frac{\partial u}{\partial y}(x_0,y_0)$, $b_2=\frac{\partial v}{\partial y}(x_0,y_0)$. (2)

• $u,v\in C^1(G)$ (u,v have continuous partial derivatives) $\Longrightarrow$ f is $\mathbb{R}$-diff. on G.

$\boxed{P_2}$ Let $f=u+iv: G\to\mathbb{C}$, $z_0=x_0+iy_0\in G$. f is $\mathbb{R}$-diff. at z_0 $\Longleftrightarrow$

$\Longleftrightarrow$ $\exists\,\alpha,\beta\in\mathbb{C}$, $\exists\,\omega: G\setminus\{z_0\}\to\mathbb{C}$ s.t. $\lim\limits_{z\to z_0}\omega(z)=0$,

$$f(z)=f(z_0)+\alpha(x-x_0)+\beta(y-y_0)+\omega(z)|z-z_0|,\quad z=x+iy\in G\setminus\{z_0\}.$$

Proof: $f=u+iv$ is $\mathbb{R}$-diff. at z_0 $\Longleftrightarrow$ $\exists\, a_1,b_1,a_2,b_2\in\mathbb{R}$, $\exists\,\omega_1,\omega_2: G\setminus\{z_0\}\to\mathbb{R}$ s.t. $\lim\limits_{(x,y)\to(x_0,y_0)}\omega_1(x,y)=\lim\limits_{(x,y)\to(x_0,y_0)}\omega_2(x,y)=0$ and

$$u(x,y)+iv(x,y)=u(x_0,y_0)+iv(x_0,y_0)+(a_1+ia_2)(x-x_0)+(b_1+ib_2)(y-y_0)+$$
$$+(\omega_1(x,y)+i\omega_2(x,y))\|(x-x_0,y-y_0)\|,\ x+iy\in G\setminus\{z_0\}$$

$$\xLeftrightarrow[\substack{\omega(z)=\omega_1(z)+i\omega_2(z)\\ f(z)=u(z)+iv(z)}]{\substack{\alpha=a_1+ia_2\\ \beta=b_1+ib_2}} f(z)=f(z_0)+\alpha(x-x_0)+\beta(y-y_0)+\omega(z)|z-z_0|,\ z=x+iy\in G\setminus\{z_0\}.$$

⚠ $\alpha=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)$, $\beta=\frac{\partial u}{\partial y}(x_0,y_0)+i\frac{\partial v}{\partial y}(x_0,y_0)$. (2')

The Cauchy-Riemann theorem

$\boxed{T}$ (the characterization of complex differentiable functions)

Let $G\subseteq\mathbb{C}$ be open, $f=u+iv: G\to\mathbb{C}$, $z_0=x_0+iy_0\in G$.

Then: f is differentiable at z_0 $\Longleftrightarrow$

$\Longleftrightarrow$ f is $\mathbb{R}$-differentiable at z_0 and

$$(*)\begin{cases}\frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0)\\ \frac{\partial u}{\partial y}(x_0,y_0)=-\frac{\partial v}{\partial x}(x_0,y_0).\end{cases}$$

$(*)$ is called the Cauchy-Riemann system (conditions) of $f=u+iv$ at z_0.

Proof: $\boxed{\Longrightarrow}$ Assume that f is diff. at z_0.

$\boxed{P1}$ with (1) $\Rightarrow \exists\, \alpha = f'(z_0)$, $\exists\, \omega : G\setminus\{z_0\} \to \mathbb{C}$ s.t. $\lim\limits_{z\to z_0} \omega(z) = 0$

and $f(z) = f(z_0) + \alpha(z - z_0) + \omega(z)|z - z_0|$, $z \in G\setminus\{z_0\}$

$$= f(z_0) + \alpha(x - x_0) + \underbrace{i\alpha}_{\beta}(y - y_0) + \omega(z)|z - z_0|.$$

$\boxed{P_2}$ with (2') $\Rightarrow$ f is $\mathbb{R}$-diff. at z_0 and

$$\left.\begin{aligned} \alpha &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) \Big| \cdot i \\ i\alpha = \beta &= \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0) \end{aligned}\right\} \Rightarrow i\frac{\partial u}{\partial x}(x_0,y_0) - \frac{\partial v}{\partial x}(x_0,y_0) = i\frac{\partial v}{\partial y}(x_0,y_0) + \frac{\partial u}{\partial y}(x_0,y_0) \Rightarrow (*).$$

$\boxed{\Longleftarrow}$ Assume that f is $\mathbb{R}$-diff. at z_0 and $(*)$ holds.

$\boxed{P_2}$ with (2') $\Rightarrow \exists\, \alpha = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$, $\exists\, \beta = \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)$,

$\exists\, \omega : G\setminus\{z_0\} \to \mathbb{C}$ s.t. $\lim\limits_{z\to z_0} \omega(z) = 0$ and

$f(z) = f(z_0) + \alpha(x - x_0) + \beta(y - y_0) + \omega(z)|z - z_0|$, $z = x + iy \in G\setminus\{z_0\}$

$$(*) \Rightarrow \beta = -\frac{\partial v}{\partial x}(x_0,y_0) + i\frac{\partial u}{\partial x}(x_0,y_0) = i\underbrace{\left(\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)\right)}_{\alpha}$$

$$\beta = i\alpha \Rightarrow f(z) = f(z_0) + \alpha(x - x_0) + i\alpha(y - y_0) + \omega(z)|z - z_0|,\ z = x + iy \in G\setminus\{z_0\}$$

$$= f(z_0) + \alpha(z - z_0) + \omega(z)|z - z_0|$$

$\Rightarrow f$ is $\mathbb{C}$-diff. at $z_0 \overset{\boxed{P1}}{\Longrightarrow} f$ is diff. at z_0.

(!) All the conditions in the C.-R. theorem are essential.

Ex1 $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$, $z \in \mathbb{C}$.

$f = u + iv \Rightarrow u(x,y) = x$,

$v(x,y) = -y$, $z = x + iy \in \mathbb{C}$.

f is $\mathbb{R}$-diff. on $\mathbb{C}$ (see Remark1), but

f is not diff. at any $z = x + iy \in \mathbb{C}$, because

$\frac{\partial u}{\partial x}(x,y) = 1 \neq \frac{\partial v}{\partial y}(x,y) = -1 \Rightarrow (*)$ does not hold.

Ex. 2 $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} + i\frac{xy}{\sqrt{x^2+y^2}}, & z = x+iy \in \mathbb{C}^* \\ 0, & z = 0 \end{cases}$,

is cont. on $\mathbb{C}$, $f = u + iv$, u, v have partial derivatives and satisfy the C.-R. sys. at $z = 0$, but f is not diff. at $z = 0$.