

THE CAUCHY-RIEMANN ALGORITHM

Let $G \subseteq \mathbb{C}$ be open, $f = u + iv : G \rightarrow \mathbb{C}$ $z_0 = x_0 + iy_0 \in G$
 $(u = \operatorname{Re} f, v = \operatorname{Im} f)$

[P] f is differentiable at $z_0 \Leftrightarrow f$ is $\mathbb{C}$ differentiable at z_0

$$(\exists \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \in \mathbb{C})$$

$(\exists \alpha \in \mathbb{C}, \exists w: G \setminus \{z_0\} \rightarrow \mathbb{C}$
 with $\lim_{z \rightarrow z_0} w(z) = 0$ s.t.

$$f(z) = f(z_0) + \alpha(z - z_0) +$$

Moreover $\alpha = f'(z_0)$

Def 1 f is R-differentiable at z_0 if u, v are FRECHET

DIFFERENTIABLE at $\underbrace{(x_0 + iy_0)}_{x_0 + iy_0}$

Ob Recall that $\mathbb{R}^2$ is identified with $\mathbb{C}: (x, y) = x + iy, \forall x, y \in \mathbb{R}$

REMARK. $\|(x, y)\| = \sqrt{x^2 + y^2} \quad \forall (x, y) \in \mathbb{C}$ (the Euclidean norm in $\mathbb{R}^2$)

• f is R-diff. at $z_0 = (x_0, y_0) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \exists a, b_1 \in \mathbb{R}, \exists w_1: G \setminus \{(x_0, y_0)\} \rightarrow \mathbb{R} \text{ s.t. } \lim_{(x, y) \rightarrow (x_0, y_0)} w_1(x, y) = 0, \\ u(x, y) = u(x_0, y_0) + a_1(x - x_0) + b_1(y - y_0) + \underbrace{w_1(x, y)}_{\|w_1(x, y)\| \cdot \|(x - x_0, y - y_0)\|}, \\ (x, y) \in G \setminus \{(x_0, y_0)\} \end{cases}$$

AND

$$\exists a_2, b_2 \in \mathbb{R}, \exists w_2: G \setminus \{(x_0, y_0)\} \rightarrow \mathbb{R} \text{ s.t. } \lim_{(x, y) \rightarrow (x_0, y_0)} w_2(x, y) = 0,$$

$$v(x, y) = v(x_0, y_0) + a_2(x - x_0) + b_2(y - y_0) + \underbrace{w_2(x, y)}_{\|w_2(x, y)\| \cdot \|(x - x_0, y - y_0)\|}, \\ (x, y) \in G \setminus \{(x_0, y_0)\}$$

REMARK 1: • $a_1 = \frac{\partial u}{\partial x}(x_0, y_0) \quad a_2 = \frac{\partial v}{\partial x}(x_0, y_0) \quad b_1 = \frac{\partial u}{\partial y}(x_0, y_0) \quad b_2 = \frac{\partial v}{\partial y}(x_0, y_0)$

• If $u, v \in C^1(G)$ (u, v , continuous functions partial derivatives) $\Rightarrow$
 $\Rightarrow f$ is R diff at G

P2 f is $\mathbb{R}$ -differentiable at $(x_0, y_0) \Leftrightarrow$ ~~there~~

$$\Leftrightarrow \exists \alpha, \beta \in \mathbb{C}, \exists w: G \setminus \{x_0, y_0\} \rightarrow \mathbb{C} \text{ with } \lim_{z \rightarrow z_0} w(z) = 0$$

$$\text{n.t. } f(z) = f(z_0) + \alpha(x-x_0) + \beta(y-y_0) + w(z)|z-z_0|, \\ \forall z = x+iy \in G \setminus \{z_0\}$$

MOREOVER $\alpha = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \quad \beta = \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0)$

PROOF f is $\mathbb{R}$ diff. at $z_0 \Leftrightarrow \exists a_1, b_1, a_2, b_2 \in \mathbb{R}$,

$$f = u + iv$$

$$\exists w_1, w_2: G \setminus \{z_0\} \rightarrow \mathbb{R}$$

$$\text{n.t. } \lim_{(x,y) \rightarrow (x_0, y_0)} w_1(x,y) = \lim_{(x,y) \rightarrow (x_0, y_0)} w_2(x,y) = 0$$

and $f(z_0)$

$$u(x,y) + i v(x,y) = u(x_0, y_0) + i v(x_0, y_0) + \overbrace{(a_1 + i a_2)(x-x_0)}^{\alpha} + \overbrace{(b_1 + i b_2)(y-y_0)}^{\beta} + \\ + \underbrace{(w_1(x,y) + i w_2(x,y))}_{w(z)} \underbrace{\|x-x_0, y-y_0\|}_{|z-z_0|}, \quad x+iy \in G \setminus \{z_0\}$$

$$\begin{aligned} \alpha &= a_1 + i a_2 \\ \beta &= b_1 + i b_2 \end{aligned}$$

$$\Rightarrow$$

$$f(z) = f(z_0) + \alpha(x-x_0) + \beta(y-y_0) + w(z)|z-z_0|, \quad z = x+iy \in G \setminus \{z_0\}$$

$$w(z) = w_1(z) + i w_2(z)$$

$$f(z) = u(z) + i v(z)$$

THE CAUCHY-RIENMANN THEOREM

T (the characterization of complex differentiable functions)

f is differentiable at $z_0 \Leftrightarrow f$ is $\mathbb{R}$ diff. at z_0 and

$$(*) \quad \begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{cases}$$

(*) is called THE CAUCHY RIENMANN SYSTEM (conditions)
of $f = u + iv$ at z_0

PROOF $\Rightarrow$ Assume that f is diff at z_0

[P1] with (1) $\Rightarrow \exists \alpha = f'(z_0)$, $\exists w: G \setminus \{z_0\} \rightarrow \mathbb{C}$ s.t. $\lim_{z \rightarrow z_0} w(z) = 0$

$$\text{and } f(z) = f(z_0) + \alpha(z - z_0) + w(z)|z - z_0|, \quad z \in G \setminus \{z_0\}$$

$$= f(z_0) + \underbrace{\alpha(x - x_0)}_{\beta} + \underbrace{i\alpha(y - y_0)}_{\beta} + w(z)|z - z_0|$$

[P2] with (2) $\Rightarrow f$ is R-diff at z_0 and

$$\left. \begin{aligned} \alpha &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) / i \\ i\alpha &= \beta = \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0) \end{aligned} \right\} \Rightarrow \underbrace{i \frac{\partial u}{\partial x}(x_0, y_0) - \frac{\partial v}{\partial x}(x_0, y_0)}_{= i \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}(x_0, y_0)} = (*)$$

$\Leftarrow$ Assume that f is R-diff. at z_0 and (*) holds

[P2] with (2) $\Rightarrow \exists \alpha = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$ $\exists \beta = \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0)$

$\exists w: G \setminus \{z_0\} \rightarrow \mathbb{C}$ s.t. $\lim_{z \rightarrow z_0} w(z) = 0$ AND

$$f(z) = f(z_0) + \alpha(x - x_0) + \beta(y - y_0) + w(z)|z - z_0| \quad \forall z = x + iy \in G \setminus \{z_0\}$$

$$(*) \Rightarrow \beta = -\frac{\partial v}{\partial x}(x_0, y_0) + i \frac{\partial u}{\partial x}(x_0, y_0) = i \left(\frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \right)$$

$$\beta = i\alpha \Rightarrow f(z) = f(z_0) + \alpha(x - x_0) + i\alpha(y - y_0) + w(z)|z - z_0| \quad \forall z = x + iy \in G \setminus \{z_0\}$$

$$= f(z_0) + \alpha(z - z_0) + w(z)|z - z_0|$$

$\Rightarrow f$ is $\mathbb{C}$ diff at z_0 [P1] $\Rightarrow f$ is diff at z_0

∇ All the conditions in the G.-R theorem are essential

Ex1: $f: \mathbb{C} \rightarrow \mathbb{C}$, $f(z) = \bar{z}$, $z \in \mathbb{C}$

$$f = u + iv \Rightarrow u(x, y) = x$$

$$v(x, y) = -y, \quad z = x + iy \in \mathbb{C}$$

f is R diff on $\mathbb{C}$ but f is NOT diff at any $z = x + iy \in \mathbb{C}$ because

$$\frac{\partial u}{\partial x}(x, y) = 1 \neq \frac{\partial v}{\partial y}(x, y) = -1 \Rightarrow * \text{ does not hold}$$

Ex 2 $f: \mathbb{C} \rightarrow \mathbb{C}$, $f(z) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} + i \frac{xy}{\sqrt{x^2+y^2}} \end{cases}$, $z = x+iy \in \mathbb{C}^*$

is cont on $\mathbb{C}$, $f = u+iv$

u, v have partial derivatives and satisfy (*) at $z=0$, but f is not differentiable at $z=0$

NOTATIONS $\frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$

$\frac{\partial f}{\partial y}(z_0) = \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0)$

REMARK From the proof of Theorem 1 we have

if f is differentiable at z_0 , then

$\alpha = f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i \frac{\partial f}{\partial y}(z_0)$
 $-i\beta \quad (\beta = i\alpha \Leftrightarrow \alpha = -i\beta)$

NOTATIONS 2 we consider the DIFFERENTIABLE OPERATORS

$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$

$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$

P3 Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be $\mathbb{R}$ -differentiable at $z_0 \in \mathbb{C}$.

Then the $\mathbb{C}$ - $\mathbb{R}$ nys (*) $\Leftrightarrow \frac{\partial f}{\partial \bar{z}}(z_0) = 0$

Moreover, if f is DIFFERENTIABLE at z_0 , then $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$

PROOF Let $f = u+iv$, $z_0 = x_0+iy_0$

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left[\left(\frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \right) + i \left(\frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0) \right) \right]$$

$$= \frac{1}{2} \left[\left(\frac{\partial u}{\partial x}(x_0, y_0) - \frac{\partial v}{\partial y}(x_0, y_0) \right) + i \left(\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0) \right) \right]$$

So, $\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \Leftrightarrow (*)$. If f is differentiable at z_0 then

$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2} \left(\frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0) \right) = f'(z_0)$

Pr

28 Q: $G \rightarrow \mathbb{C}$ is R-diff at $z_0 \in G$, then

~~28~~ $\exists w: G \setminus \{z_0\} \rightarrow \mathbb{C}$ n.t. $\lim_{z \rightarrow z_0} w(z) = 0$ and

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0) \cdot (z - z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \overline{(z - z_0)} + w(z) \quad | z - z_0| + z \in G \setminus \{z_0\}$$