CURS 5
ANALIZĂ

Examples of entire functions

1. THE COMPLEX POLYNOMIAL FUNCTION

$p: \mathbb{C} \to \mathbb{C}$, $p(z) = a_0 + a_1 z + \dots + a_m z^m$, $z \in \mathbb{C}$
where $a_0, \dots, a_m \in \mathbb{C}$. There $p \in \mathcal{H}(\mathbb{C})$ and
$$p'(z) = a_1 + 2a_2 z + \dots + m a_m z^{m-1}, \quad z \in \mathbb{C}$$

2. THE COMPLEX EXPONENTIAL FUNCTION

Sem 4: if $z = x + iy \in \mathbb{C}$, then
$\exists \lim_{m \to \infty} \left(1 + \frac{z}{m}\right)^m = e^z$, where $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

$\exp: \mathbb{C} \to \mathbb{C}^*$, $\exp(z) = e^z$, $z \in \mathbb{C}$ is the COMPLEX EXPONENTIAL FCT.

REMARK: $\exp \in \mathcal{H}(\mathbb{C})$ and $(e^z)' = e^z$, $z \in \mathbb{C}$

PROOF: $\exp = u + iv \Rightarrow u(x,y) = e^x \cos y$
$v(x,y) = e^x \sin y$ $\quad x + iy \in \mathbb{C}$

So $u, v \in C^1(\mathbb{R}^2) \Rightarrow \exp$ is $\mathbb{R}$ diff on $\mathbb{C}$

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = e^x \cos y = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) = e^x(-\sin y) = -\frac{\partial v}{\partial x}(x,y) \end{cases}$$
$\Rightarrow$ exp. satisfies the C-R syst $\forall z = x + iy \in \mathbb{C} \Rightarrow$ $\exp \in \mathcal{H}(\mathbb{C})$

$$(e^z)' = \frac{\partial}{\partial x}(e^x(\cos y + i \sin y)) = e^x(\cos y + i \sin y) = e^z, \quad z = x + iy \in \mathbb{C}$$

REMARK
- $|e^z| = e^x$, $\arg(e^z) = y \pmod{2\pi}$ $\forall z = x + iy \in \mathbb{C}$
- $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$ $\quad z_1, z_2 \in \mathbb{C}$ $\quad e^{-z} = \frac{1}{e^z}$, $z \in \mathbb{C}$
- $e^{z + 2k\pi i} = e^z \cdot \underbrace{e^{2k\pi i}}_{1} = e^z$, $\forall z \in \mathbb{C}$, $k \in \mathbb{Z}$

  So exp is $(2\pi i)$ - periodic on $\mathbb{C}$
  ($\Rightarrow$ exp is not injective on $\mathbb{C}$)
- $z = x \in \mathbb{R} \Rightarrow \exp(z) = e^x$ (real exponential fct at $x \in \mathbb{R}$)

- $\begin{cases} e^{iy} = e^0(\cos y + i\sin y) \\ e^{-iy} = e^0(\cos y - i\sin y) \end{cases} \Rightarrow \begin{cases} \cos y = \dfrac{e^{iy}+e^{-iy}}{2} \\ \sin y = \dfrac{e^{iy}-e^{-iy}}{2i} \end{cases}$ (EULER'S FORMULA) $y \in \mathbb{R}$

3. THE COMPLEX TRIGONOMETRIC FUNCTIONS

$\cos, \sin : \mathbb{C} \to \mathbb{C}$

$$\boxed{\begin{aligned} \cos z &= \frac{e^{iz}+e^{-iz}}{2}, \quad z \in \mathbb{C} \\ \sin z &= \frac{e^{iz}-e^{-iz}}{2i}, \quad z \in \mathbb{C} \end{aligned}}$$

REMARK
- $\cos, \sin \in \mathcal{H}(\mathbb{C})$

  $(\cos z)' = -\sin z \qquad (\sin z)' = \cos z \qquad z \in \mathbb{C}$

- $\cos(z) = \cos z$ $\quad \forall z \in \mathbb{C}$

  $\sin(-z) = -\sin z$

- $\cos(z + 2k\pi) = \cos z$ $\quad \forall z \in \mathbb{C}, \forall k \in \mathbb{Z}$
- $\sin(z + 2k\pi) = \sin z$

4. THE COMPLEX HYPERBOLIC FUNCTIONS

$ch, sh : \mathbb{C} \to \mathbb{C}$, $\quad ch\, z = \dfrac{e^z + e^{-z}}{2} \qquad sh\, z = \dfrac{e^z - e^{-z}}{2}, \quad z \in \mathbb{C}$

REMARK
- $ch, sh \in \mathcal{H}(\mathbb{C})$ $\qquad (ch\, z)' = sh\, z$ $\qquad z \in \mathbb{C}$

  $(sh\, z)' = ch\, z$

- $ch^2 z - sh^2 z = 1 \quad \forall z \in \mathbb{C}$