

CRITERION FOR HOLOMORPHIC FUNCTIONS TO BE CONSTANT

Recall: $D \subseteq \mathbb{C}$ is a domain, if D is open and connected

$\nabla f \in \mathcal{H}(G), G \subseteq \mathbb{C}$ open $\Rightarrow f \in C(G)$ (continuous on G)

1. Let $\emptyset \neq D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D), f = u + iv$

The following are equivalent: i) f is constant on D

ii) $f' \equiv 0$ on D

iii) $\operatorname{Re} f$ is constant on D

iv) $\operatorname{Im} f$ is constant on D

v) $|f|$ is constant on D

PROOF: i) $\Rightarrow$ ii iii iv v is clear

ii $\Rightarrow$ i Assume that $f' \equiv 0$ on D . Fix $z_0 \in D$ and $E = \{z \in D: f(z) = f(z_0)\}$

• E is closed on D : if $(z_n)_{n \in \mathbb{N}}$ in E s.t. $z = \lim_{n \rightarrow \infty} z_n \in D$, then $z \in E$,

because $f \in \mathcal{H}(D) \Rightarrow f \in C(D) \Rightarrow \lim_{n \rightarrow \infty} f(z_n) = f(z)$
 $\Rightarrow f(z_0) = f(z) \Rightarrow z \in E$

• E is open in D : Let $a \in E$. Since $a \in D, \exists n > 0$ s.t. $U(a, n) \subseteq D$.
 Let $b \in U(a, n)$

Then $[a, b] = \{(1-t)a + tb: t \in [0, 1]\} \subset U(a, n)$

Let $g: [0, 1] \rightarrow D, g(t) = (1-t)a + tb, t \in [0, 1]$

g is differentiable at any $t_0 \in (0, 1)$ and $g([0, 1]) = [a, b]$

Let $g = f \circ g$. For $t, t_0 \in [0, 1], t \neq t_0$

$$\frac{g(t) - g(t_0)}{t - t_0} = \frac{f(z(t)) - f(z(t_0))}{z(t) - z(t_0)} \cdot \frac{z(t) - z(t_0)}{t - t_0}$$

$$\text{So, } \lim_{t \rightarrow t_0} \frac{g(t) - g(t_0)}{t - t_0} = \frac{f \in \mathcal{H}(D)}{\lim_{t \rightarrow t_0} z(t) = z(t_0)} \underbrace{f'(z(t_0)) \cdot (b-a)}_{=0} = 0 \quad \forall t_0 \in (0,1)$$

$$\Rightarrow g' \equiv 0 \text{ on } (0,1) \Rightarrow g \text{ is constant on } [0,1]$$

$$\Rightarrow g(0) = g(1) \Rightarrow f(a) = f(b) \quad \left. \begin{array}{l} a \in E \Rightarrow f(a) = f(z_0) \\ b \in E \Rightarrow f(b) = f(z_0) \end{array} \right\} \Rightarrow f(b) = f(z_0) \Rightarrow b \in E$$

So $U(a, \pi) \subseteq E$. We deduce that E is open in D .

E is open and closed in $D \xRightarrow{\text{connected}} E = D \text{ or } E = \emptyset$

$$\left. \begin{array}{l} E = D \text{ or } E = \emptyset \\ z_0 \in E \Rightarrow E \neq \emptyset \end{array} \right\} \Rightarrow E = D \Rightarrow f \text{ is constant in } D$$

iii) $\Rightarrow$ i Assume that $u = \operatorname{Re} f$ is constant on D .

$$\text{Then } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \equiv 0 \text{ on } D \xRightarrow{\text{ii} \Rightarrow \text{i}} f \text{ is constant on } D.$$

iv) $\Rightarrow$ i is proved as above

v) $\Rightarrow$ i Assume that $|f|$ is constant on D . Then $\exists c > 0$ s.t. $u^2 + v^2 = c$ on D

$$\bullet c = 0 \Rightarrow u = v \equiv 0 \Rightarrow f \equiv 0$$

$$\bullet c > 0. \text{ Let } z = x + iy \in D$$

$$u^2 + v^2 = c \Rightarrow \frac{\partial}{\partial x} (u^2 + v^2)(x, y) = \frac{\partial}{\partial y} (u^2 + v^2)(x, y) = 0$$

$$\Rightarrow \left\{ \begin{array}{l} 2u(x, y) \frac{\partial u}{\partial x}(x, y) + 2v(x, y) \frac{\partial v}{\partial x}(x, y) = 0 \\ 2u(x, y) \frac{\partial u}{\partial y}(x, y) + 2v(x, y) \frac{\partial v}{\partial y}(x, y) = 0 \end{array} \right\} \text{C.R.} \Rightarrow \left\{ \begin{array}{l} u(x, y) \frac{\partial u}{\partial x}(x, y) + v(x, y) \frac{\partial v}{\partial x}(x, y) \\ v(x, y) \frac{\partial u}{\partial x}(x, y) + (-u(x, y)) \frac{\partial v}{\partial x}(x, y) \end{array} \right.$$

$$\begin{vmatrix} u(x, y) & v(x, y) \\ v(x, y) & -u(x, y) \end{vmatrix} = -u^2(x, y) - v^2(x, y) = -c \neq 0 \Rightarrow$$

the above lin. homogeneous syst. has a unique sol

$$\left(\frac{\partial u}{\partial x}(x, y), \frac{\partial v}{\partial x}(x, y) \right) = (0, 0)$$

$$\infty f' = \frac{\partial f}{\partial x} = 0 \xRightarrow{i \Rightarrow i} f \text{ is constant on } D$$

C. Let $D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$

If $f(D) \subseteq \mathbb{R}$ or $f(D) \subseteq i\mathbb{R}$, then f is constant on D

$$\text{PROOF: } \left. \begin{array}{l} f(D) \subseteq \mathbb{R} \Rightarrow \operatorname{Im} f = 0 \\ \text{or} \\ f(D) \subseteq i\mathbb{R} \Rightarrow \operatorname{Re} f = 0 \end{array} \right\} \xRightarrow{\tau_1} f \text{ is constant on } D$$

HARMONIC FUNCTIONS

$$\nabla^2 \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \text{ is the LAPLACE operator in } \mathbb{R}^n$$

D. Let $G \subseteq \mathbb{R}^n$ be open. $u: G \rightarrow \mathbb{R}$ is HARMONIC

$$\text{if } u \in C^2(G) \text{ and } \Delta u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \quad \forall x + iy \in G$$

(the LAPLACE EQUATION)

[P1] Let $G \subseteq \mathbb{C}$ open and $f \in \mathcal{H}(G)$, $f = u + iv$

Then $u = \operatorname{Re} f$, $v = \operatorname{Im} f$ are harmonic on G .

PROOF: $\forall f \in \mathcal{H}(G) \Rightarrow u, v \in C^\infty(G)$ (this will be proved in a future course)

$$G: \mathbb{R} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial v}{\partial x} \right) =$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \quad \underline{v \in C^2(G)} = 0 \quad \Rightarrow u \text{ is harmonic}$$

Similarly, v is harmonic

Ex: $u: \mathbb{C} \rightarrow \mathbb{R}$, $u(x,y) = e^x \cos y$, $x+iy \in \mathbb{C}$

$$\Rightarrow u = \operatorname{Re} f \quad \text{where } f(z) = e^z = e^x (\cos y + i \sin y)$$

$$z = x+iy \in \mathbb{C}$$

$\Rightarrow u$ is harmonic on $\mathbb{C}$

$v = \operatorname{Im} f$ is harmonic on $\mathbb{C}$

② If $G \subseteq \mathbb{C}$ is open, $u: G \rightarrow \mathbb{R}$ is harmonic on G , then

$\exists f \in \mathcal{H}(G)$ s.t. $u = \operatorname{Re} f$?

[D₂] Let $G \subseteq \mathbb{C}$ be open and $P, Q: G \rightarrow \mathbb{R}$ be s.t. $P, Q \in C^1(G)$.

Then: $\omega = P dx + Q dy$ is called a (LINEAR) DIFFERENTIAL FORM

of class C^1 on G

$\forall z \in G: \omega(z) = P(z) dx + Q(z) dy$ is a linear function from $\mathbb{R}^2$ to $\mathbb{R}$, where $dx(x,y) = x$, $dy(x,y) = y$, $(x,y) \in \mathbb{R}^2$

ω is closed if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on G

• exact if $\exists f: G \rightarrow \mathbb{R}$, $f \in C^2(G)$ s.t.

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y} \quad (\text{i.e. } df = \omega)$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$