

HARMONIC FUNCTIONS

[P₁] (POINCARÉ) Let $D \subseteq \mathbb{R}^2$ be a starlike domain w.r.t. $z_0 \in D$ and w be a differential form of class C^1 on D .

Then: w is closed $\Leftrightarrow$ w is exact

[T₁] Let $D \subseteq \mathbb{C}$ be a starlike domain w.r.t. $z_0 \in D$ and $u: D \rightarrow \mathbb{R}$ be HARMONIC. Then $\exists f \in \mathcal{H}(D)$ s.t. $u = \operatorname{Re} f$.

PROOF: We look for $v: D \rightarrow \mathbb{R}$, $v \in C^1(D)$ s.t. u and v satisfy the Cauchy-Riemann eqs on D :

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

on D , since this implies $f = u + iv \in \mathcal{H}(D)$, by the Cauchy Riemann theorem.

Let $P = -\frac{\partial u}{\partial y}$, $Q = \frac{\partial u}{\partial x}$ on D

u is harmonic on $D \Rightarrow u \in C^2(D) \Rightarrow w = Pdx + Qdy$ is a diff. form of class C^1 on D

w is closed: $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on $D \Leftrightarrow \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial y} \right)$ on D

$\Leftrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on D

$\Leftrightarrow \Delta u = 0$ on D , which is true because u is harmonic on D

D is a starlike domain w.r.t. $z_0 \in D$

$\xRightarrow{P_1} w$ is exact $\Rightarrow \exists v: D \rightarrow \mathbb{R}$ $v \in C^1(D)$ s.t.

$P = \frac{\partial v}{\partial x}$ and $Q = \frac{\partial v}{\partial y}$ ($w = dv$) $\Rightarrow$

$$\Rightarrow \begin{cases} -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \end{cases} \quad (*) \quad f = u + iv \quad u, v \in C^2(D) \Rightarrow f \text{ is } \mathbb{R} \text{ differentiable on } D$$

* $\Rightarrow f$ satisfies the C.R. eqs. on $D \Rightarrow f \in \mathcal{H}(D)$

[C1] Let $D \subseteq \mathbb{C}$ be a starlike domain w.r.t. $z_0 \in D$ and $v: D \rightarrow \mathbb{R}$ be harmonic. Then $\exists f \in \mathcal{H}(D)$ s.t. $v = \operatorname{Im} f$

PROOF T1. $\Rightarrow \exists g \in \mathcal{H}(D)$ s.t. $v = \operatorname{Re} g$

Let $f = ig$ $f \in \mathcal{H}(D)$ and $\operatorname{Im} f = v$

THE COMPLEX MULTIVALUED LOGARITHM

Let $w \in \mathbb{C}^*$. The eq. $z^z = w$ has infinitely many solutions:

$$e^z = w \xrightarrow{z=x+iy} e^x (\cos y + i \sin y) = |w| (\cos(\arg w) + i \sin(\arg w))$$

$$\Leftrightarrow \begin{cases} e^x = |w| > 0 \\ y = \arg w + 2k\pi, k \in \mathbb{Z} \end{cases} \quad \Leftrightarrow z_k = \ln|w| + i(\arg w + 2k\pi), k \in \mathbb{Z}$$

D1. Let $\operatorname{Log}: \mathbb{C}^* \rightarrow \mathcal{P}(\mathbb{C})$

$$\operatorname{Log} z = \{ \ln|z| + i(\arg z + 2k\pi) : k \in \mathbb{Z} \} = \ln|z| + i \operatorname{Arg} z \quad z \in \mathbb{C}^*$$

Log is the (complex) MULTIVALUED LOGARITHM

[R1] 1. $w \in \operatorname{Log} z \Leftrightarrow e^w = z \quad (\forall z \in \mathbb{C}^*, w \in \mathbb{C})$

2. $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 \quad \forall z_1, z_2 \in \mathbb{C}^*$

3. $\operatorname{Log} \frac{1}{z} = -\operatorname{Log} z, \forall z \in \mathbb{C}^*$

[D1] Let $D \subseteq \mathbb{C}^*$ be a domain. $f: D \rightarrow \mathbb{C}$ is a BRANCH of Log on D , if $f \in \mathcal{H}(D)$ and $f(z) \in \operatorname{Log} z, \forall z \in D$

P₂ Let $D \subseteq \mathbb{C}^*$ be a domain.

i) If f is a branch of $\log$ on D , then $f'(z) = \frac{1}{z} \quad \forall z \in D$

ii) If f_1, f_2 are branches of $\log$ on D , then $\exists k_0 \in \mathbb{Z}$ s.t.

$$f_2(z) = f_1(z) + 2k_0\pi i \quad \forall z \in D$$

Moreover, if in addition $\exists z_0 \in D$ s.t. $f_1(z_0) = f_2(z_0)$, then

$$f_1(z) = f_2(z) \quad \forall z \in D$$

PROOF: i) $f \in \mathcal{H}(D)$ and $e^{f(z)} = z \quad \forall z \in D$

$$\Rightarrow (e^{f(z)})' = e^{f(z)} \cdot f'(z) = 1 \quad \forall z \in D$$

$$\Rightarrow f'(z) = \frac{1}{z} \quad \forall z \in D$$

$$\Rightarrow f_1'(z) = f_2'(z) = \frac{1}{z} \quad \forall z \in D$$

$$\Rightarrow (f_2 - f_1)' = 0 \text{ on } D, D \text{ is domain}$$

T₁, Corollary $\Rightarrow f_2 - f_1$ is constant on D

$$f_1, f_2 \in \mathcal{H}(D) \Rightarrow \theta_2 - \theta_1 \text{ is constant on } D$$

$$\Rightarrow k \text{ is constant on } D$$

$$\Rightarrow \exists k_0 \in \mathbb{Z} \text{ s.t. } f_2(z) = f_1(z) + 2k_0\pi i \quad \forall z \in D$$

$$\text{If } \exists z_0 \in D \text{ s.t. } f_1(z_0) = f_2(z_0) \Rightarrow k_0 = 0 \Rightarrow f_2 = f_1 \text{ on } D$$

P₃ Let $B = \{z \in \mathbb{C} : -\pi < \text{Im } z < \pi\}$

$$D = \mathbb{C} \setminus (-\infty, 0] = \{w \in \mathbb{C} : \text{Re } w \leq 0, \text{Im } w = 0\}$$

$$\text{and } f: B \rightarrow D \quad f(z) = \exp(z) = e^z, z \in B$$

Then f is bijective on B and $f^{-1}(w) = \ln|w| + i \arg w, w \in D$

PROOF: From Serimmar7 we have:

$$B_\alpha = \{z \in \mathbb{C} : \alpha \leq \text{Im } z < \alpha + 2\pi\} \quad \alpha \in \mathbb{R}$$

$$L_a = \{z \in \mathbb{C} : \text{Re } z = a\} \parallel O_y, a \in \mathbb{R}$$

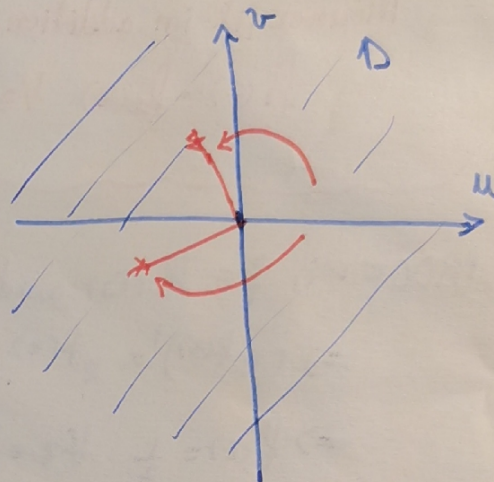
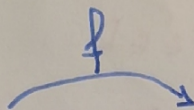
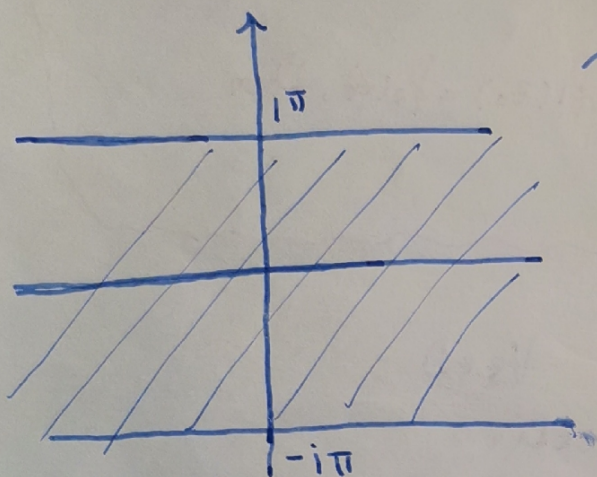
$$L_b = \{z \in \mathbb{C} : \text{Im } z = b\} \parallel O_x, b \in \mathbb{R}$$

$\exp$ is injective on B , $\exp(B) = \mathbb{C}^*$,

$$\exp(d_0) = \exp(0, e^0)$$

$$\exp(d_\pi) = \{w \in \mathbb{C}^* : \arg w = \pi \pmod{2\pi}\}$$

$$B = B_{-\pi} \setminus \tilde{d}_{-\pi}$$



$$\begin{aligned} f \text{ is injective on } B \text{ and } f(B) &= \mathbb{C}^* \setminus \exp(\tilde{d}_{-\pi}) = \\ &= \mathbb{C}^* \setminus \{w \in \mathbb{C}^* : \arg w = -\pi \pmod{2\pi}\} \\ &= \mathbb{C}^* \setminus (-\infty, 0) \\ &= \mathbb{C} \setminus (-\infty, 0] = D \end{aligned}$$

$$\text{Let } w \in D. f(z) = w, z \in B \Leftrightarrow e^z = w, z \in B$$

$$\Leftrightarrow e^{\operatorname{Re} z} (\cos \operatorname{Im} z + i \sin \operatorname{Im} z) = |w| (\cos(\arg w) + i \sin(\arg w)) \neq 0$$

$$\Leftrightarrow \begin{cases} \operatorname{Re} z = \ln |w| > 0 \\ \operatorname{Im} z = \arg w \in (-\pi, \pi) \end{cases} \Leftrightarrow z = \ln |w| + i \arg w$$

$$\text{So } f^{-1}(w) = \ln |w| + i \arg w, \forall w \in D$$

$$D_2 \text{ Let } D = \mathbb{C} \setminus (-\infty, 0], B = \{w \in \mathbb{C} : -\pi < \operatorname{Im} w < \pi\}$$

$$\log : D \rightarrow B, \log z = \ln |z| + i \arg z, z \in D \text{ is}$$

the PRINCIPAL BRANCH of $\log$ on D

$\boxed{P_2}$ • $\log \in \mathcal{H}(D)$ (mult. meromorphic) and

$\& \log 1 = 0 \xrightarrow{P_2} \log$ is the unique branch of Log on D
s.t. $\log 1 = 0$

• $\boxed{P_2} \Rightarrow (\log z)' = \frac{1}{z}, z \in D$

• $z = x \in (0, \infty) \Rightarrow \log x = \ln x$. So $\log|_{(0, \infty)} = \ln$

$\boxed{P_3}$ • $\forall k \in \mathbb{Z} : \log_k : D \rightarrow \mathbb{C}$ $\log_k = \log + 2k\pi i$
is the unique branch of Log on D
s.t. $\log_k 1 = 2k\pi i$

• $\log_k(D) = 2k\pi i + B, \forall k \in \mathbb{Z}$