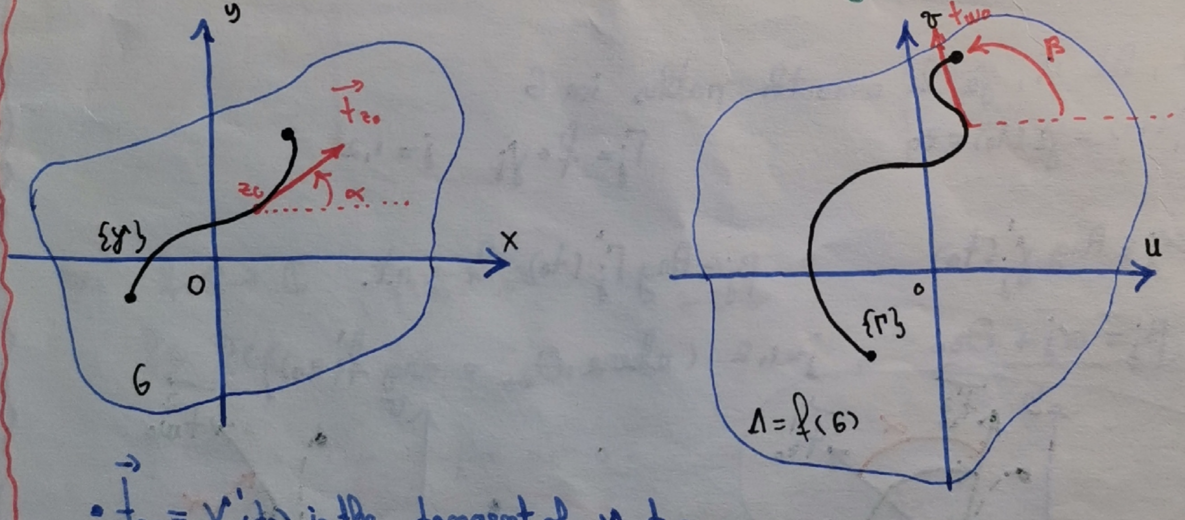
CURS 8
ANALIZĂ

THE GEOMETRIC INTERPRETATION OF THE COMPLEX DERIVATIVE

Let $G \subseteq \mathbb{C}$ be open, $a, b \in \mathbb{R}$, $a < b$

D1 Let $\gamma : [a, b] \to G$ be a continuous function.
γ is a PATH in G. $\{\gamma\} = \gamma([a, b])$ is SUPPORT of γ
γ is SMOOTH, if $\gamma \in C^1([a, b])$ and $\gamma'(t) \neq 0, \forall t \in [a, b]$

- Let $f \in \mathcal{H}(G)$ be s.t. $f'(z) \neq 0$, $\forall z \in G$
  $\Delta = f(G)$ and fix $z_0 \in G$ $w_0 = f(z_0) \in \Delta$
- Let γ be a smooth path in G s.t. $z_0 \in \{\gamma\}$
  $\Rightarrow \exists t_0 \in [a, b]$ s.t. $\gamma(t_0) = z_0$
- Let $\Gamma = f \circ \gamma \Rightarrow \Gamma$ is a smooth path in Δ s.t. $w_0 \in \{\Gamma\}$,
  $w_0 = f(\gamma(t_0))$, and $\Gamma'(t_0) = f'(z_0) \cdot \gamma'(t_0)$ *



- $\vec{t}_{z_0} = \gamma'(t_0)$ is the tangent of γ at z_0
  $\vec{t}_{w_0} = \Gamma'(t_0)$ is the tangent of Γ at w_0

$\boxed{R_1}$ $(*) \Longrightarrow |\Gamma'(t_0)| = |f'(z_0)| \cdot |\gamma'(t_0)|$

$K_{z_0} = |f'(z_0)| \Longrightarrow |\vec{t}_{w_0}| = K_{z_0} \cdot |\vec{t}_{z_0}|$

↳ is the COEFFICIENT OF LINEAR DEFORMATION at z_0 and it is independent of γ

So, f contracts/expands by the factor K_{z_0} the tangent vectors at z_0, if $K_0 \neq 1$

- If $K_0 < 1$ then we have a <u>CONTRACTION</u> at z_0
  $\{z \in G : |f'(z)| < 1\}$ is the set that is contracted by f
- If $K_{z_0} > 1$, then we have an <u>EXPANSION</u> at z_0
  $\{z \in G : |f'(z)| > 1\}$ is the set that is expanded by f

$\boxed{R_2}$ Let $\theta_{z_0} = \arg f'(z_0) \overset{*}{\Longrightarrow} \operatorname{Arg} \Gamma'(t_0) = \theta_{z_0} + \operatorname{Arg} \gamma'(t_0) \Longrightarrow$

$\Longrightarrow \underbrace{\arg \vec{t}_{w_0}}_{\beta} = \theta_{z_0} + \underbrace{\arg \vec{t}_{z_0}}_{\alpha} \pmod{2\pi}$

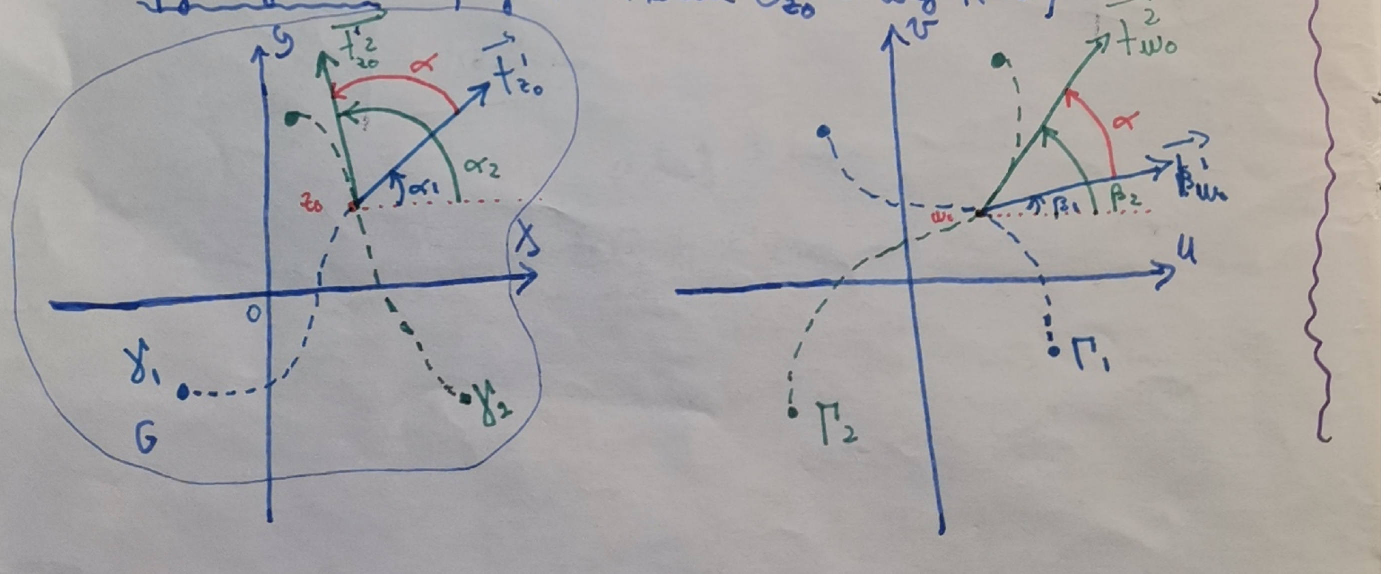
So θ_{z_0} is the <u>ANGLE OF ROTATION</u> of $\vec{t}_{z_0}$ through f.

Next, let γ_1, γ_2 be smooth paths in G

$\gamma_1(t_0) = \gamma_2(t_0) = z_0$ $\qquad \Gamma_j = f \circ \gamma_j \quad j = 1,2$

Let $\alpha_j \in \operatorname{Arg} \gamma_j'(t_0)$ $\qquad \beta_j \in \operatorname{Arg} \Gamma_j'(t_0)$ be s.t.

<u>$\beta_j = \alpha_j + \theta_{z_0}$</u>, $j = 1,2$ (where $\theta_{z_0} = \arg f'(z_0)$)

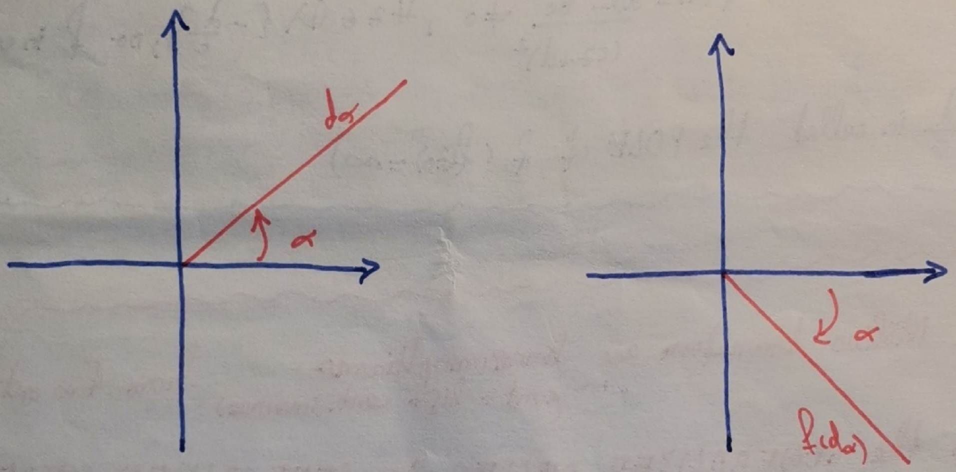


$$\beta_2-\beta_1=\alpha_2-\alpha_1 \Rightarrow \angle(\underset{\gamma_1'(t_0)}{\underset{\parallel}{\vec{t}^{\,1}_{z_0}}}, \underset{\gamma_2'(t_0)}{\underset{\parallel}{\vec{t}^{\,2}_{z_0}}}) = \angle(\underset{\Gamma_1'(t_0)}{\underset{\parallel}{\vec{t}^{\,1}_{w_0}}}, \underset{\Gamma_2'(t_0)}{\underset{\parallel}{\vec{t}^{\,2}_{w_0}}}) \pmod{2\pi}$$

So $f \in \mathcal{H}(G)$ with $f'(z) \neq 0 \ \forall z \in G$ preserves the measures and the orientation of the angles between smooth paths. Such a map is called CONFORMAL

Ex1 $f: \mathbb{C} \to \mathbb{C}$ $f(z) = \bar{z}$, $z \in \mathbb{C}$

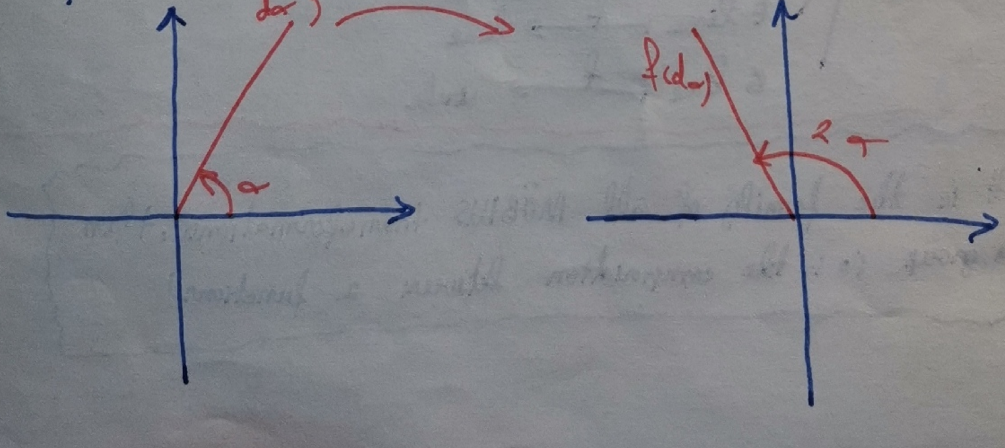
f preserves the measures of angles, but NOT the orientation



Ex2 $f: \mathbb{C} \to \mathbb{C}$ $f(z) = z^2$ $z \in \mathbb{C}$

$f \in \mathcal{H}(\mathbb{C})$ but $f'(0) = 0$

f doubles the angles at 0



LINEAR FRACTIONAL / MÖBIUS TRANSFORMATIONS

D_2 Let $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$, $f(z) = \dfrac{az+b}{cz+d}$, $z \in \mathbb{C}_\infty$,

where $a, b, c, d \in \mathbb{C}$ satisfy $\underline{ad - bc \neq 0}$

R_1 • $\boxed{c=0} \Rightarrow f(z) = \frac{a}{d} z + \frac{b}{d}$, $z \in \mathbb{C}_\infty$. So, $\lim_{z\to\infty} f(z) = f(\infty) = \infty$

Also $f \in \mathcal{H}(\mathbb{C})$ and $f'(z) = \frac{a}{d}$, $z \in \mathbb{C}$. Since $\frac{a}{d} \neq 0$,

f is conformal.

$\boxed{c \neq 0} \Rightarrow \lim_{z\to\infty} f(z) = f(\infty) = \frac{a}{c}$, $f \in \mathcal{H}(\mathbb{C} \setminus \{-\frac{d}{c}\})$,

$f'(z) = \dfrac{ad - bc}{(cz+d)^2} \neq 0$, $\forall z \in \mathbb{C} \setminus \{-\frac{d}{c}\}$, so f is conformal

$z_0 = -\frac{d}{c}$ is called the POLE of f ($f(z_0) = \infty$)

$\boxed{P_1}$ The Möbius transform are homeomorphisms (cont + bij + cont. inverse) from $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$

that map the GENERALIZED CIRCLES to GENERALIZED CIRCLES

Let $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$, $f(z) = \dfrac{az+b}{cz+d}$, $z \in \mathbb{C}_\infty$

• $c = 0$ circle $\xmapsto{f}$ circle and line $\xmapsto{f}$ line

• $c \neq 0$ $z_0 = -\frac{d}{c}$

- $\in$ circle $\xmapsto{f}$ line
- $\notin$ circle $\xmapsto{f}$ circle
- $\in$ line $\xmapsto{f}$ line
- $\notin$ line $\xmapsto{f}$ circle

R_2 If H is the family of all MÖBIUS transformations, then $(H, \circ)$ is a group ($\circ$ is the composition between 2 functions)

D_3 Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct

$(z_1, z_2, z_3, z_4) = \dfrac{z_1 - z_2}{z_1 - z_4} : \dfrac{z_3 - z_2}{z_3 - z_4}$ is the CROSS-RATIO of z_1, z_2, z_3, z_4

P_2 MÖBIUS transform. preserve the cross-ratio of any distinct points in $\mathbb{C}_\infty$.

$f \in H$ and $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ distinct $\Rightarrow (z_1, z_2, z_3, z_4) = (f(z_1), f(z_2), f(z_3), f(z_4))$

$\boxed{R_3}$ If $z_1, z_2, z_3 \in \mathbb{C}_\infty$ are distinct and $w_1, w_2, w_3 \in \mathbb{C}_\infty$ are distinct then $\exists!\ f \in H$ s.t. $f(z_j) = w_j,\ j \in \{1, 2, 3\}$

f is obtained by solving the eq.:

$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3) \Rightarrow w = f(z)$ is the sol