# Seminar 13

$\gamma(t) = z_0 + r \cdot e^{it} \Rightarrow \{\gamma\} = U(z_0, r)$

① Compute the following integrals:

1) $\int \cos \frac{z-1}{z} dz$ $\quad \gamma(t) = e^{it}; \ t \in [0, 2\pi]$

2) $\varphi_r = \int_{\gamma_r} \frac{1}{(z^2+1)^3} dz; \ \gamma_r(t) = r \cdot e^{it}; \ t \in [0, 2\pi]$
where $r \in (0,1) \cup (1, \infty)$

3) $\int_\gamma \frac{e^{\frac{1}{z}} + e^{\frac{1}{z-1}}}{z} dz; \ \gamma(t) = e^{it}; \ t \in [0, 2\pi]$

4) $\int_\gamma \frac{\sin z}{z \cdot \cos z} dz; \ \gamma(t) = \frac{1}{2} + \frac{3}{2} e^{it}$

**Theorem**: $f \in \mathcal{H}(G)$, $S$ - sing points; $S' \cap G = \emptyset$

$\gamma$ a Jordan-contour: $\gamma \underset{G}{\simeq} 0$.

$\Rightarrow \boxed{\int_\gamma f = 2\pi \cdot i \cdot \sum \operatorname{Res}(f, z_0)}$

$\operatorname{Res}(f, z_0) = \text{coef of } \frac{1}{z - z_0}$

$e^\varphi = 1 + \frac{1}{1!}\varphi + \frac{1}{2!}\varphi^2 + \ldots + \frac{1}{n!}\varphi^n$

$\sin \varphi = \varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 - \ldots$

$\cos \varphi = 1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 - \ldots$

1) $f(z) = \cos\frac{z-1}{z} \Rightarrow f : \mathbb{C}^* \to \mathbb{C}$ → residue only at $z_0$

I $\cos\frac{z-1}{z} = \frac{1}{z}(\cos z - 1) = \frac{1}{z}\left(\cancel{1-1} - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \dots\right)$

$= -\frac{1}{2!}z + \frac{1}{4!}z^3 - \dots$

↳ coef of $\frac{1}{z-z_0} = \frac{1}{z}$ is $0$ ($\frac{1}{z}$ not here)

$\Rightarrow \int_\gamma f = 2\pi i \cdot \sum \mathrm{Res}(f, z_0) = 2\pi \cdot i \cdot 0 = 0$

II $\lim_{z\to 0} \cos\frac{z-1}{z} = \lim_{z\to 0} \frac{\cos z - \underbrace{\cos 0}_{0}}{z} = \cos' 0 = -\sin 0 = 0$

→ $z_0$ removable $\Rightarrow \mathrm{Res}(f, 0) = 0 \Rightarrow \int_\gamma f = 0$

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1') $\int_\gamma \frac{z}{e^z - 1}\, dz$ ; $\gamma(t) = e^{it}$

$e^z = 1 \Rightarrow e^{x+iy} = 1 \Leftrightarrow e^{x+iy} = e^0(\cos 0 + i \cdot \sin 0) \Rightarrow$

$\Rightarrow z_k = i(2k\pi)$

$\begin{cases} z_{-1} = -2\pi i \\ z_0 = 0 \\ z_1 = 2\pi i \end{cases}$ $\Big\}$ $\Rightarrow z_0 = 0 \in \mathrm{int}\,\gamma$

$\lim_{z\to z_0} \frac{z}{e^z - 1} = \lim_{z\to z_0} \frac{z - 0}{e^z - e^0} = \frac{1}{(e^z)'} = 1 \Rightarrow z_0$ removable $\Rightarrow$

$\Rightarrow \mathrm{Res}(f, 0) = 0 \Rightarrow$

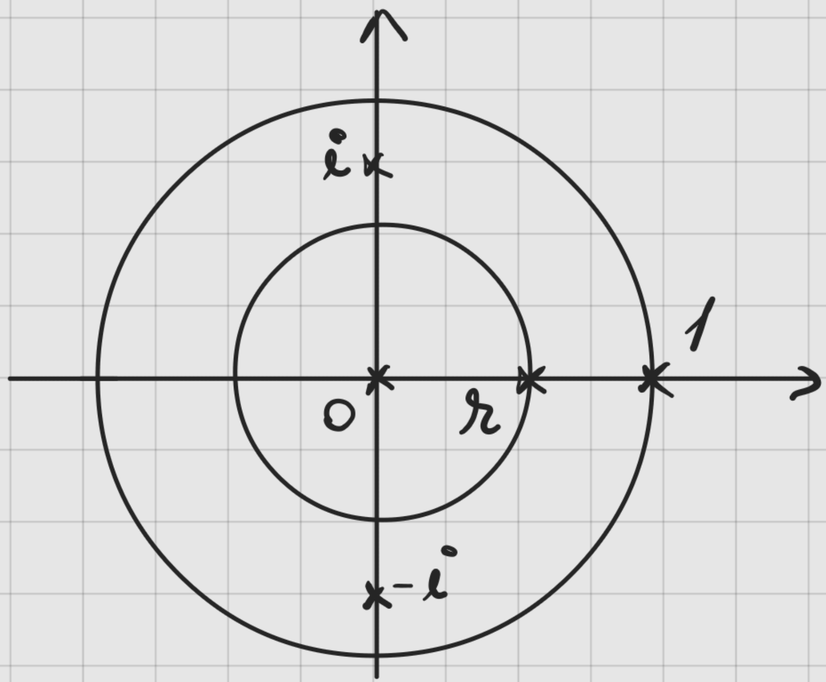
$\Rightarrow \int_\gamma f = 0$

2) $\int_{\gamma_r} = \int_{\gamma_r} \frac{1}{(z^2+1)^3} dz$ $\qquad z^2+1=0 \Leftrightarrow z_{1,2} = \pm i$

case $r \in (0,1)$ : $\int_\gamma f = 0$ because we have no sing p.

$r \in (1,\infty) \Rightarrow z = i;\ z = -i \in \text{int}\ \gamma \Rightarrow$

$\Rightarrow \int_\gamma f = 2\pi i \left[ \text{Res}(f, i) + \text{Res}(f, -i) \right]$




• $\text{Res}(f, i) = ?$ $\qquad \lim_{z \to i} f(z) = \frac{1}{0^+} = \infty \Rightarrow z_0 = i$
pole

$f(z) = \frac{g}{h}$

$g(z) = 1$ $\qquad g, h \in \mathcal{H}(U(i, ?)) = \mathcal{H}(U(i, 1))$

$h(z) = (z^2+1)^3$

$h'(z) = 3(z^2+1)^2 \cdot 2z = 6z(z^2+1)^2 = 0$

$$h''(z) = 6(z^2+1)^2 + 6z \cdot 2(z^2+1) \cdot 2z = 6(z^2+1)^2 + 24z^2(z^2+1) = 0$$

$$h'''(z) = 24z(z^2+1) + 24 \cdot 2z(z^2+1) + 24z^2 \cdot 2z \neq 0 \Rightarrow$$

$\Rightarrow$ order of $h(z)$ is 3

$$\text{Res}(f, i) = \frac{1}{(3-1)!} \lim_{z \to i} \left[(z-i)^3 \cdot f(z)\right]^{(2)} =$$

$$= \frac{1}{2!} \lim_{z \to i} \left[(z-i)^3 \cdot \frac{1}{(z+1)^3}\right]^{(2)} = \frac{1}{2!} \lim_{z \to i} \left[\left(\frac{z-i}{z+1}\right)^3\right]^{(2)} =$$

$$= \frac{1}{2} \lim_{z \to i} \left[\left(\frac{z-i}{(z-i)(z+i)}\right)^3\right]^{(2)} =$$

$$= \frac{1}{2} \lim_{z \to i} \left[\left(\frac{1}{z+i}\right)^3\right]'' = \frac{1}{2} \lim_{z \to i} \frac{-4}{(z+i)^5} = \frac{1}{2} \cdot \frac{1}{(i+1)^5} = \frac{3}{16i}$$

$$\left(\left(\frac{1}{z+i}\right)^3\right)' = 3\left(\frac{1}{z+i}\right)^2 - \frac{1}{(z+i)^2} = \frac{-3}{(z+i)^4}$$

$$\left(\frac{-3}{(z+i)^4}\right)' = \frac{12}{(z+i)^5}$$

$-$ analog $-i \Rightarrow \text{Res}(f, -i) = \dfrac{-3}{16i} \Rightarrow$

$$\Rightarrow \int_\gamma f = 2\pi i \left(\frac{3}{16i} - \frac{3}{16i}\right) = 0$$

3) $\int_\gamma \frac{e^{\frac{1}{z}}}{z}dz$ ; $\gamma(t) = e^{it}$ ; $t \in [0, 2\pi]$

$z_0 = 0$ the only isol sing point of $f$.

$Res(f, 0) = ?$

$$\frac{e^{\frac{1}{z}}}{z} = \frac{1}{z}\left(1 + \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \dots\right) =$$

$$= \frac{1}{z} + \frac{1}{1!}\cdot\frac{1}{z^2} + \frac{1}{2!}\cdot\frac{1}{z^3} - \dots$$

$\hookrightarrow Res(f, 0) =$ coef of $\frac{1}{z-0} = 1$

$$= \int_\gamma f(z) = 2\pi i \sum \underbrace{Res(f, 0)}_{1} = 2\pi i$$