Complex

Analysis

LECTURE 1

3. X. 2023

Final grade:
- 40% seminar activity:
  - 10% activity
  - 30% mid term exam (courses 1-6)
- 60% final exam: - from all courses (No theory)

Complex numbers

Consider the following operation on $\mathbb{R}^2$:

$$\boxed{\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}}$$

$(\mathbb{R}^2, +, \cdot)$ is a commutative field, which we denote $\mathbb{C}$.

$(\mathbb{R}\times\{0\}, +, \cdot)$ is a subfield of $\mathbb{C}$, $\varphi: \mathbb{R} \to \mathbb{R}\times\{0\}$ given by $\varphi(x) = (x, 0)$ is an isomorphism w.r.t. the usual addition and multiplication on $\mathbb{R}$.

- We identify $\mathbb{R}$ with $\mathbb{R}\times\{0\}$. So $\mathbb{R}\subseteq\mathbb{C}$.

<u>Denote</u>: $i=(0,1)$, called the imaginary unit

$$\forall (x,y)\in\mathbb{C},\quad (x,y)=(x,0)+(0,y)$$
$$=x+(y,0)\cdot(0,1)$$
$$=x+y\cdot i$$

$$\forall z\in\mathbb{C},\ \exists!\ x\in\mathbb{R},\ \exists!\ y\in\mathbb{R} \text{ s.t. } \boxed{z=x+i\cdot y}$$

<u>Remark</u>:

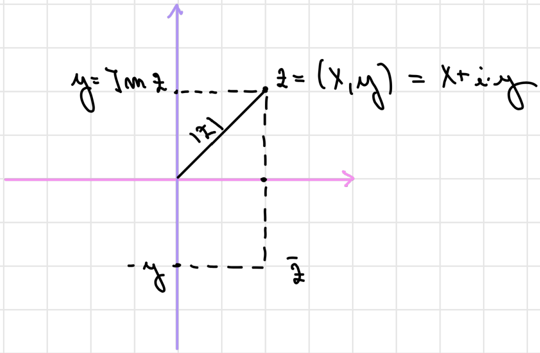
$$i^2=i\cdot i=(0,1)\cdot(0,1)=(0-1,0+0)=(-1,0)=-1$$

<u>Remark</u>:

$\mathbb{C}$ is NOT an <u>ordered</u> field! (cannot define "<", ">")

<u>The algebric form</u>:

Let $z=x+i\cdot y\in\mathbb{C}$. Denote: $\operatorname{Re} z=x$ (real part), $\operatorname{Im} z=y$ (imaginary part)



- $|z| = \sqrt{x^2 + y^2}$
- $\bar{z} = x - iy$
  ↳ conjugate

Properties:

- $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$
- $\operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$
- $|z| = |\bar{z}|$
- $|z^2| = z \cdot \bar{z}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, $\forall z_1, z_2 \in \mathbb{C}$
- $|z_1 + z_2| \leq |z_1| + |z_2|$

The trigonometric form:

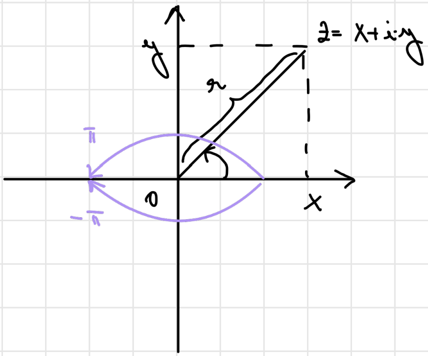
Let $z \in \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C}\setminus\{0\} = \mathbb{C}\setminus\{0,0\}$

Denote $|z| = r > 0$.

Every $\theta \in \mathbb{R}$ that satisfies

(*) $z = r(\cos\theta + i\cdot\sin\theta)$

is called an argument of z.



Remark:

$z=0$ has NO argument.

$\theta_0 = \arg z$ = the angle between $\vec{O_z}$ and the positive real semiaxis

$$\begin{cases} \cos\theta_0 = \dfrac{x}{r} \\ \sin\theta_0 = \dfrac{y}{r} \end{cases}$$

$\exists!\ \theta_0 \in [-\pi, \pi]$ s.t. (*) holds, called the principal argument, denoted $\boxed{\theta_0 = \arg z}$

$\text{Arg } z := \{\arg z + 2k\pi \mid k \in \mathbb{Z}\}$ is the set of all arguments of z.

$\text{Arg}: \mathbb{C}^* \longrightarrow \mathcal{P}(\mathbb{R})$ is called the multivalued arguement function.

Properties:

$$\operatorname{Arg}(z_1 \cdot z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

$$(\Leftrightarrow \operatorname{Arg}(z_1 \cdot z_2) = \arg z_1 + \arg z_2 \pmod{2\pi})$$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2, \quad \forall z_1, z_2 \in \mathbb{C}^*$$

$$\operatorname{Arg} \frac{1}{z} = -\operatorname{Arg} z = \operatorname{Arg} \bar{z}, \quad z \in \mathbb{C}^*$$

$$\operatorname{Arg}(r \cdot z) = \operatorname{Arg} z, \quad z \in \mathbb{C}^*, \ r \in (0, \infty)$$

Remark:

$\forall z \in \mathbb{C}^*$, $\boxed{z = r(\cos\theta + i \cdot \sin\theta) \quad r = |z|, \ \theta \in \operatorname{Arg} z}$ is the trigonometric form of z.

Topology of the complex plane

$\forall P(x,y) \in \mathbb{R}^2$ is identified with the complex number $z = x + iy$

The complex plane is the set of complex numbers with the Euclidian structure of $\mathbb{R}^2$, which we denote again by $\mathbb{C}$.

If $z_1 = x_1 + i \cdot y_1$, $z_2 = x_2 + i \cdot y_2$, the Euclidian distance between z_1 and z_2 is $\boxed{|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)}}$

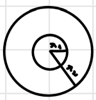
$U(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \}$ open disk

$\bar{U}(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| \le r \}$ closed disk

$\partial U(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| = r \}$ circle

$\dot{U}(z_0, r) = \{ z \in \mathbb{C} \mid 0 < |z - z_0| < r \}$ punctured disk

$U(z_0, r_1, r_2) = \{ z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2 \}$ the annulus



Let $A \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$.

Recall from Calculus in $\mathbb{R}^n$:

z is an:

- a is an interior point of A if $A \in \mathcal{V}(a)$, i.e., $\exists\, r > 0$ s.t. $B(a, r) \subseteq A$.
- a is an adherent / closure point of A if $\forall\, V \in \mathcal{V}(a)$, $V \cap A \neq \emptyset$
- a is an accumulation / a limit point of A if $\forall\, V \in \mathcal{V}(a)$, $V \cap A \setminus \{a\} \neq \emptyset$
- a is a boundary point of A if $\forall\, V \in \mathcal{V}(a)$, $V \cap A \neq \emptyset$ and $V \cap (\mathbb{R}^n \setminus A) \neq \emptyset$
- a is an isolated point of A if $\exists\, V \in \mathcal{V}(a)$ s.t. $V \cap A = \{a\}$

A is:

- open, if $\forall\, z \in A$ is an interior point of A
- closed, if $\forall\, z \in A$ is a closure point of A
- bounded, if $\exists\, r > 0$ s.t. $A \subseteq U(z_0, r)$

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- $z_1, z_2 \in \mathbb{C}^*$

$$z_j = r_j(\cos\theta_j + i\sin\theta_j),\quad r_j = |z_j|,\ \theta_j \in \operatorname{Arg} z_j,\ j = 1,2$$

$$z_1 z_2 = (r_1 r_2)(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

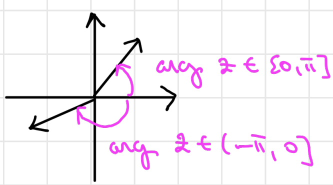
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1+\theta_2))$$

- $z \in \mathbb{C}^*,\ n \in \mathbb{N}$

$z = r(\cos\theta + i\sin\theta),\ r = |z| > 0;\ \theta \in \operatorname{Arg} \rightarrow$ the trigonometric form of z

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

$$\operatorname{Arg} z = \{\underbrace{\arg z}_{\in(-\pi,\pi)} + 2k\pi : k \in \mathbb{Z}\}$$



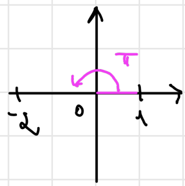
10% GRADE → answering at the seminar

1. Write the trigonometric form of: $1, -2, i, -i, 1+i, 1-i,$

$1-\sqrt{3}\cdot i,\ \dfrac{1+i}{1-i},\ \left(\dfrac{1+i}{1-i}\right)^2$

- $1 = 1\cdot(\cos\theta + i\sin\theta)$, $\theta \in \operatorname{Arg} 1 = \{0 + 2k\pi,\ k\in\mathbb{Z}\}$

$|1| = 1$



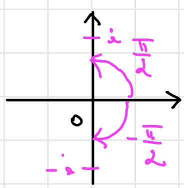
$1 = \cos(2k\pi) + i\sin(2k\pi),\ k\in\mathbb{Z}$

- $-2 = 2\cdot\left(\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\right),\ k\in\mathbb{Z}$

$|-2| = 2$

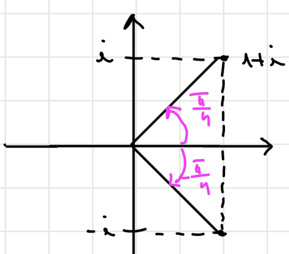
- $i = \cos\left(\frac{\pi}{2}+2k\pi\right) + i\cdot\sin\left(\frac{\pi}{2}+2k\pi\right),\ k\in\mathbb{Z}$

$|i| = 1$



- $-i = \cos\left(-\frac{\pi}{2}+2k\pi\right) + i\sin\left(-\frac{\pi}{2}+2k\pi\right),\ k\in\mathbb{Z}$

- $1+i = \sqrt{2}\left(\underbrace{\cos\left(\frac{\pi}{4}+2k\pi\right)}_{\frac{1}{\sqrt{2}}} + i\cdot\sin\underbrace{\left(\frac{\pi}{4}+2k\pi\right)}_{\frac{1}{\sqrt{2}}}\right),\ k\in\mathbb{Z}$

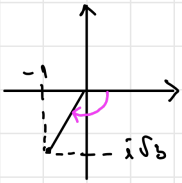


$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}+2k\pi\right) + i\sin\left(-\frac{\pi}{4}+2k\pi\right)\right),\ k\in\mathbb{Z}$

- $-1-\sqrt{3}\cdot i = 2\cdot\left(\overbrace{\cos\left(-\frac{2\pi}{3}+2k\pi\right)}^{-\frac{1}{2}} + i\cdot\overbrace{\sin\left(-\frac{2\pi}{3}+2k\pi\right)}^{-\frac{\sqrt{2}}{2}}\right),\ k\in\mathbb{Z}$

$$|-1-\sqrt{3}\cdot i| = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{4} = 2$$



- $\frac{1+i}{1-i} = \frac{(i+2)^2}{2} = \frac{\cancel{1^2}+\cancel{i^2}+2i}{2} = i = \cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right)$

$1+i$

$z\cdot\overline{z} = |z|^2$

↳ conjugate

$$\frac{1+i}{1-i} = \frac{\cancel{\sqrt{2}}\left(\cos\left(\frac{\pi}{4}+2k_1\pi\right)+i\sin\left(\frac{\pi}{4}+2k_1\pi\right)\right)}{\cancel{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}+2k_2\pi\right)+i\sin\left(-\frac{\pi}{4}+2k\pi\right)\right)} =$$

$$= \cos\left(\frac{\pi}{2}+2k\pi\right)+i\cdot\sin\left(\frac{\pi}{2}+2k\pi\right),\ k\in\mathbb{Z}$$

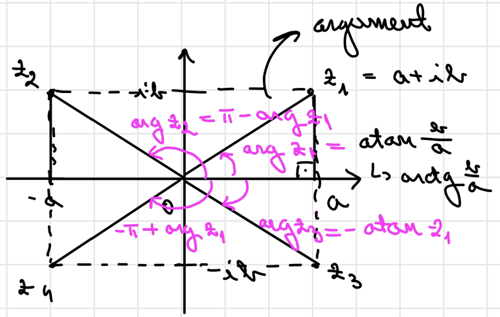
- $\underbrace{\left(\frac{1+i}{1-i}\right)^2}_{i^2=-1} = 1^2\cdot\left(\cos\left(\cancel{2}\cdot\frac{\pi}{\cancel{2}}+2k\pi\right)+i\cdot\sin\left(\cancel{2}\cdot\frac{\pi}{\cancel{2}}+2k\pi\right)\right),\ k\in\mathbb{Z}$

2. $a, b > 0$, $z_1 = a + ib$

$z_2 = -a + ib$

$z_3 = a - ib$

$z_4 = -a - ib$



$\operatorname{Arg} z_j = ?$; $j = \overline{1,4}$

$\operatorname{Arg} z_1 = \{ \operatorname{atan}\frac{b}{a} + 2k\pi,\ k \in \mathbb{Z} \}$

$\operatorname{Arg} z_2 = \{ \pi - \operatorname{atan}\frac{b}{a} + 2k\pi,\ k \in \mathbb{Z} \}$

$\operatorname{Arg} z_3 = \{ -\operatorname{atan}\frac{b}{a} + 2k\pi,\ k \in \mathbb{Z} \}$

$\operatorname{Arg} z_4 = \{ -\pi + \operatorname{atan}\frac{b}{a} + 2k\pi,\ k \in \mathbb{Z} \}$

3. Find all the roots of order $n \in \mathbb{N}^*$ of $w \in \mathbb{C}^*$.

$z^n = w \rightarrow z = ?$

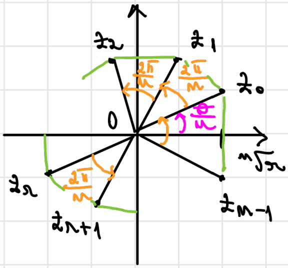
- $w \in \mathbb{C}^* \Rightarrow w = r(\cos\theta + i\sin\theta)$, $\theta \in \operatorname{Arg} w$, $r = |w| > 0$

- $z \neq 0 \Rightarrow z = \rho(\cos\alpha + i \cdot \sin\alpha)$, $\rho = |z| > 0$, $\alpha \in \operatorname{Arg} z$

$z^n = w \Leftrightarrow \rho^n(\cos(n\alpha) + i\sin(n\alpha)) = r(\cos\theta + i \cdot \sin\theta)$

$$\Leftrightarrow \begin{cases} \rho^n = r \\ n\alpha = \theta \pmod{2\pi} \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt[n]{r} \\ n\alpha = \theta + 2k\pi \end{cases},\ k \in \mathbb{Z}$$

$$\Leftrightarrow z = \sqrt[n]{r} \cdot (\cos\alpha + i \cdot \sin\alpha),\ \alpha \in \left\{ \frac{\theta}{n} + \frac{2k\pi}{n},\ k \in \mathbb{Z} \right\}$$



$$z_0 = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2\cdot 0\cdot\pi}{n}\right) + i\cdot\sin\left(\frac{\theta}{n} + \frac{2\cdot 0\cdot \pi}{n}\right)\right)$$

$$z_1 = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi}{n}\right) + i\cdot\sin\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)\right)$$

$$z_2 = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{4\pi}{n}\right) + i\cdot\sin\left(\frac{\theta}{n} + \frac{4\pi}{n}\right)\right)$$

Notation:

$$\boxed{\sqrt[n]{w} = \left\{\sqrt[n]{r}\cdot\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\cdot\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right), k = \overline{0, n-1}\right\}}$$

↳ roots of order n of w

$= \{z_0, z_1, \dots, z_{n-1}\}$

4. $\sqrt[3]{\sqrt{3}+i} = ?$, $\sqrt{\dfrac{1+i}{1-i}} = ?$

ex: $\sqrt{-1} = \{i, -i\}$

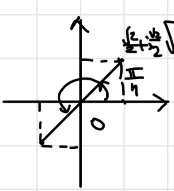
- $w = \sqrt{3} + i = 2\cdot\left(\underbrace{\cos\left(\frac{\pi}{6}\right)}_{\frac{\sqrt{3}}{2}} + i\cdot\underbrace{\sin\left(\frac{\pi}{2}\right)}_{\frac{1}{2}}\right)$

$|\sqrt{3}+i| = \sqrt{3+1} = 2$

$$\sqrt[3]{w} = \sqrt[3]{\sqrt{3}+i} = \left\{\sqrt[3]{2}\cdot\left(\cos\left(\frac{\frac{\pi}{6}}{3} + \frac{2k\pi}{3}\right) + i\cdot\sin\left(\frac{\pi}{18} + \frac{2k\pi}{3}\right)\right), k = 0,1,2\right\}$$

- $w = \dfrac{1+i}{1-i} = i = 1\cdot\left(\cos\left(\frac{\pi}{2}\right) + i\cdot\sin\left(\frac{\pi}{2}\right)\right)$

↓ pb1.



$$\sqrt{w} = \sqrt{i} = \left\{\cos\left(\frac{\frac{\pi}{2}}{2} + \frac{\cancel{2}k\pi}{\cancel{2}}\right) + i\cdot\sin\left(\frac{\pi}{4} + k\pi\right), k = 0,1\right\}$$

$$= \left\{\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right\}$$

5. a) $A = \{z \in \mathbb{C},\ |z-i| = 1\}$

b) $B = \{z \in \mathbb{C},\ |z+1-i| \le 2\}$

c) $C = \{z \in \mathbb{C},\ |z+1-i| > 2\}$

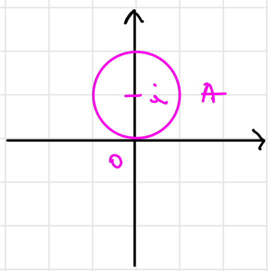
d) $D = \{z \in \mathbb{C}^*,\ \arg z = \frac{\pi}{3}\}$

e) $E = \{z \in \mathbb{C}\setminus\{1-i\},\ \arg(z-1+i) = \frac{\pi}{3}\}$

(H): 6. a) $|z+1| + |z^2+1| + |z^3+1| > 2,\ \forall\, z \in \partial U(0,1)$

b) $\left|z + \frac{1}{z}\right| \ge 2,\ z \in \mathbb{C}^* \Rightarrow \left|z^3 + \frac{1}{z^3}\right| \ge 2$

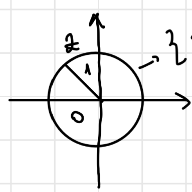
a) $A = \{z \in \mathbb{C},\ |z-i| = 1\} = \partial U(i,1)$



$\cancel{\sqrt{a^2} = a \Rightarrow |a| > 0}$

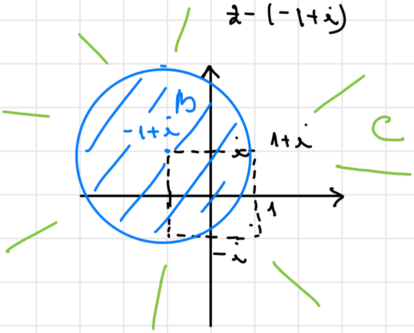
$\cancel{|a| = 1 \Rightarrow a = \pm 1}$

$|1| = |-1| = |i| = |-i| = 1$



$\{z \in \mathbb{C},\ |z| = 1\}$

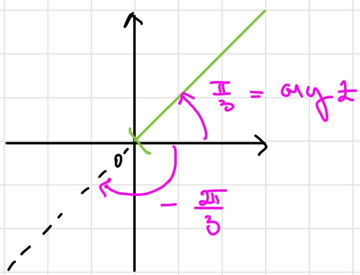
b) $B = \{ z \in \mathbb{C},\ |\underbrace{z+1-i}_{z-(-1+i)}| \leq 2 \} = \overline{D}(-1+i, 2)$



c) $C = \{ z \in \mathbb{C},\ |z+1-i| > 2 \}$

$= \mathbb{C} \setminus B$

d) $\Lambda = \{ z \in \mathbb{C}^*,\ \arg z = \frac{\pi}{3} \}$



SEMINAR 2

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1. Represent graphically the sets:

a) $A = \{z \in \mathbb{C},\ 1 < |z+2-i| < 2\}$

b) $B = \{z \in \mathbb{C} \setminus \{-1\},\ \operatorname{Im} \frac{z-1}{z+1} = 0\}$

c) $C = \{z \in \mathbb{C} \setminus \{-1\},\ \operatorname{Re} \frac{z-1}{z+1} = 0\}$

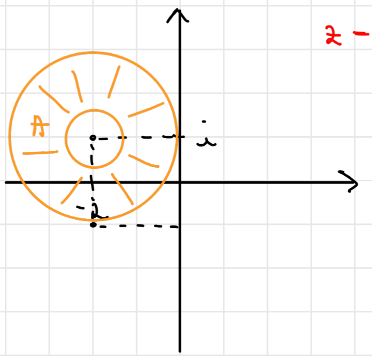
d) $\Delta = \{z \in \mathbb{C},\ (1+i)\bar{z} + (1-i)z - 1 = 0\}$

centered radius 2

a) $A = \{z \in \mathbb{C},\ 1 < |z+2-i| < 2\} = U(\underline{-2+i}, 1, 2)$

radius 1

$z = \operatorname{Re} z + i \cdot \operatorname{Im} z$



b) $B = \{z \in \mathbb{C} \setminus \{-1\},\ \operatorname{Im} \frac{z-1}{z+1} = 0\} = \{z \in \mathbb{C} \setminus \{-1\},\ \operatorname{Im} z = 0\}$

$z \in \mathbb{C} \setminus \{-1\}$ $\qquad = \mathbb{R} \setminus \{-1\}$

$$^{\bar{z}+1)}\frac{z-1}{z+1} = \frac{(\bar{z}+1)(z-1)}{|z+1|^2} = \frac{\bar{z}\cdot z + z - \bar{z} - 1}{|z+1|^2} = \frac{|z|^2 + 2i\operatorname{Im} z - 1}{|z+1|^2} =$$

$\boxed{z \cdot \bar{z} = |z|^2}$

$$= \frac{|z|^2 - 1}{|z+1|^2} + i \cdot \frac{2\operatorname{Im} z}{|z+1|^2}$$

c) $C = \{ z \in \mathbb{C} \setminus \{-1\},\ \mathrm{Re}\,\frac{z-1}{z+1} = 0 \} = \{ z \in \mathbb{C} \setminus \{-1\},\ \frac{|z|^2 - 1}{|z+1|^2} = 0 \}$

$z = x + i \cdot y$

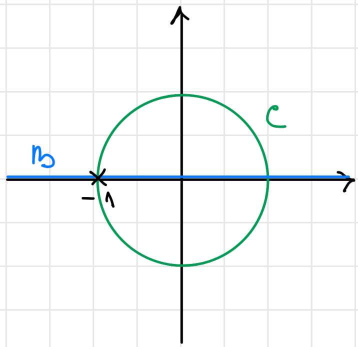
$\bar{z} = x - iy$

$z - \bar{z} = 2iy$

$y = \mathrm{Im}\, z$

$= \{ z \in \mathbb{C} \setminus \{-1\},\ |z| = 1 \}$

$= \partial U(0,1) \setminus \{-1\}$



$\partial U(z_0, r) = \{ z \in \mathbb{C},\ |z - z_0| = r \}$

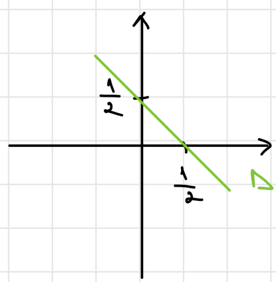
eq. line: $ax + by + c = 0$, $(a,b) \neq (0,0)$

d) $\Delta = \{ z \in \mathbb{C},\ (1+i)\bar{z} + (1-i)z - 1 = 0 \}$

$= \{ x + iy \in \mathbb{C},\ (1+i)(x - iy) + (1-i)(x+iy) - 1 = 0 \}$

$= \{ x + iy \in \mathbb{C},\ 2x + 2y - 1 = 0 \}$

$= \{ x + iy \in \mathbb{C},\ x + y = \frac{1}{2} \}$



2. Let $A, C \in \mathbb{R}$, $B \in \mathbb{C}$ s.t. $|B|^2 > AC$.

Prove that $A|z|^2 + \bar{B} \cdot z + B \cdot \bar{z} + C = 0$ $(*)$ is the eq. of either

$\underset{(A=0)}{\text{a line}}$ or $\underset{(A \neq 0)}{\text{a circle}}$.

I. $A = 0$

Let $z = x + iy$ and $B = \alpha + i\beta$

$(*) \iff (\alpha - i\beta) \cdot (x + iy) + (\alpha + i\beta)(x - iy) + C = 0$

$\iff \alpha x + \cancel{\alpha \cdot iy} - \cancel{i\beta x} + \beta y + \alpha x - \cancel{\alpha \cdot iy} + \cancel{i\beta x} + \beta y + C = 0$

$\iff 2\alpha x + 2\beta y + C = 0 \iff \alpha x + \beta y = -\frac{C}{2}$

If $\alpha = \beta = 0$, then $B = 0 \Rightarrow |B|^2 = 0 > A \cdot C = 0$, contradiction.

So, $(\alpha, \beta) \neq (0, 0) \Rightarrow$ the eq. of a line

II. $A \neq 0$

$|z - z_0| = r \iff |z - z_0|^2 = r^2$

$\iff (z - z_0)(\bar{z} - \bar{z}_0) = r^2$

$\iff \underbrace{z \cdot \bar{z}}_{|z|^2} - \bar{z} \cdot z_0 - \bar{z}_0 \cdot z + \underbrace{\underbrace{z_0 \cdot \bar{z}_0}_{|z_0|^2} - r^2}_{\in \mathbb{R}} = 0$

$$\circledast \overset{|:A}{\iff} |z|^2 + \frac{\bar{B}}{A}\cdot z + \frac{B}{A}\cdot \bar{z} + \frac{C}{A} = 0$$

$$\iff \underbrace{z\cdot\bar{z} + \left(\frac{\bar{B}}{A}\right) z + \left(\frac{B}{A}\right)\bar{z} + \left(\frac{B}{A}\right)\cdot\left(\frac{\bar{B}}{A}\right)}_{\left(z + \frac{B}{A}\right)\left(\bar{z} + \left(\frac{\bar{B}}{A}\right)\right)} - \left(\frac{B}{A}\right)\left(\frac{\bar{B}}{A}\right) + \frac{C}{A} = 0$$

$$\iff \left|z + \frac{B}{A}\right|^2 = \left|\frac{B}{A}\right|^2 - \frac{C}{A} = \frac{|B|^2}{A^2} - \frac{CA}{A^2} =$$

$$= \frac{|B|^2 - AC}{A^2}$$

$$\circledast\ z \in \partial U\left(-\frac{B}{A}, \sqrt{\frac{|B|^2 - AC}{A^2}}\right)$$

↳ correct, because $|B|^2 > AC$

3. Prove that the stereographic projection of a circle on the Riemann sphere is either a circle or a line.

$Re\, z = \frac{z + \bar{z}}{2}$

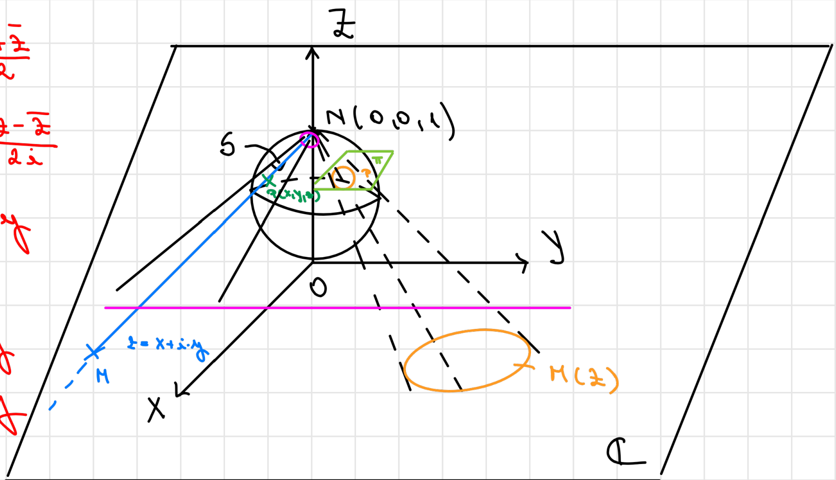
$Im\, z = \frac{z - \bar{z}}{2i}$

$z = x + i\cdot y$

$Re\, z = x$

$Im\, z = y$

$\bar{z} = x - i\cdot y$



$$(S):\ (X-0)^2 + (Y-0)^2 + \left(Z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$(S):\ X^2 + Y^2 + Z^2 - Z = \frac{1}{2}$$

$$\boxed{\begin{array}{l} \varphi : \mathbb{C} \longrightarrow S \setminus \{N\} \quad (**) \\[1em] \underset{z \in \mathbb{C}}{\varphi(z)} = \left(\underbrace{\frac{\operatorname{Re} z}{|z|^2+1}}_{X}, \underbrace{\frac{\operatorname{Im} z}{|z|^2+1}}_{Y}, \underbrace{\frac{|z|^2}{|z|^2+1}}_{Z} \right) \in S \setminus \{N\} \\[2em] \underbrace{\varphi^{-1}(X, Y, Z)}_{\in \mathbb{C}} = \dfrac{X + iY}{1 - Z}, \quad (X, Y, Z) \in S \setminus \{N\} \end{array}}$$

stereographic projection

Let $\mathscr{C} = S \cap \Pi$, where $(\Pi): aX + bY + cZ + d = 0$,

$(a, b, c, d \in \mathbb{R},\ a^2 + b^2 + c^2 \neq 0)$

$$\underset{P \neq N}{P(X, Y, Z)} \in \mathscr{C} \underset{P \in S \setminus \{N\}}{\overset{(**)}{\Longrightarrow}} \begin{cases} X = \dfrac{\operatorname{Re} z}{|z|^2+1} \\[1em] Y = \dfrac{\operatorname{Im} z}{|z|^2+1} \\[1em] Z = \dfrac{|z|^2}{|z|^2+1} \end{cases}$$

$$\overset{P \in \Pi}{\Longrightarrow} a \cdot \frac{\operatorname{Re} z}{|z|^2+1} + b \cdot \frac{\operatorname{Im} z}{|z|^2+1} + c \cdot \frac{|z|^2}{|z|^2+1} + d = 0 \ \Big| \cdot (|z|^2+1)$$

$$P \in \mathscr{C} \setminus \{N\} \Longleftrightarrow \underline{a \cdot \operatorname{Re} z + b \cdot \operatorname{Im} z + c \cdot |z|^2 + d(|z|^2+1) = 0}$$

$$\iff \underbrace{(c+d)}_{A\in\mathbb{R}}|z|^2 + \underbrace{\frac{a-i\cdot b}{2}}_{\overline{B}}\cdot z + \underbrace{\frac{a+ib}{2}}_{B}\cdot \overline{z} + \underbrace{d}_{C\in\mathbb{R}} = 0$$

We use problem 2

We need only to prove : $|B|^2 > AC$

$\pi \cap S \neq \emptyset$ $\quad dist(\pi, C) < \frac{1}{2} \iff \cdots \iff$ Homework

SEMINAR 3

23. X. 2023

1. Recall that $d_c : \mathbb{C}_\infty \times \mathbb{C}_\infty \to [0, \infty)$ is the chordal distance on $\mathbb{C}_\infty$.

a) Prove that $d_c(z_1, z_2) \in [0,1]$, $\forall z_1, z_2 \in \mathbb{C}_\infty$

b) Represent graphically the sets

$$A = \{ z \in \mathbb{C} : d_c(1+i, z) = d_c(-1-i, z) \}$$

$$B = \{ z \in \mathbb{C} : d_c(-i, z) = d(z, \infty) \}$$

$$d_c(z_1, z_2) = \begin{cases} \dfrac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, & z_1, z_2 \in \mathbb{C} \\[2ex] \dfrac{1}{\sqrt{1+|z|^2}}, & \begin{bmatrix} z_1 = z \in \mathbb{C} \\ z_2 = \infty \end{bmatrix} \text{ or } \begin{bmatrix} z_1 = \infty \\ z_2 = z \in \mathbb{C} \end{bmatrix} \\[2ex] 0, & z_1 = z_2 = \infty \end{cases}$$

$$\boxed{z \cdot \bar{z} = |z|^2}$$

a) $$\frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} \le 1, \quad \forall z_1, z_2 \in \mathbb{C}$$

$$\iff |z_1 - z_2| \le \sqrt{(1+|z_1|^2)(1+|z_2|^2)} \quad |(\ )^2$$

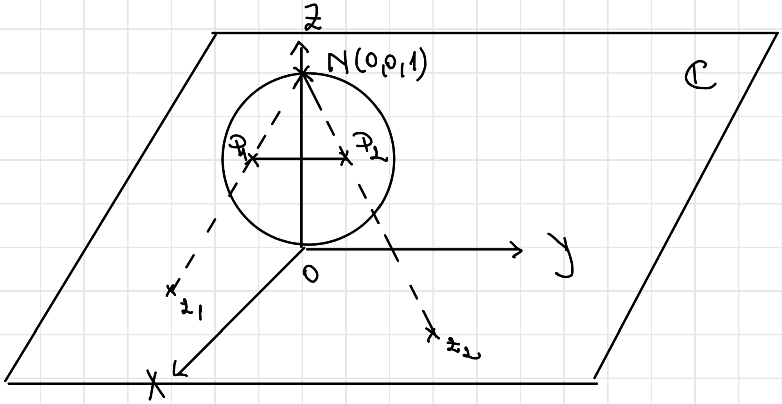
$$\iff |z_1 - z_2|^2 \le (1+|z_1|^2)(1+|z_2|^2)$$

$$\iff (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \le (1 + z_1 \cdot \bar{z}_1)(1 + z_2 \cdot \bar{z}_2)$$

$\iff \cancel{z_1\bar{z}_1} + \cancel{z_2\bar{z}_2} - z_1\bar{z}_2 - \bar{z}_1 z_2 \le 1 + \cancel{z_1\bar{z}_1} + \cancel{z_2\bar{z}_2} + (z_1\bar{z}_2)(\bar{z}_1 z_2)$

$$\iff 0 \le \underbrace{1 + z_1\bar{z}_2 + \bar{z}_1 z_2 + (z_1\bar{z}_2)(\bar{z}_1 z_2)}_{(1+z_1\bar{z}_2)\big(\underbrace{1+\bar{z}_1 z_2}_{\overline{1+z_1\bar{z}_2}}\big)}$$

$\iff 0 \le |1 + z\bar{z}_2|^2$, true

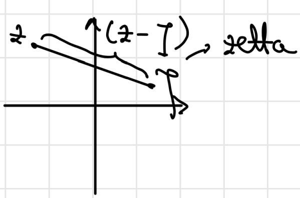


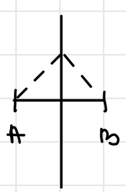
$d_{\mathbb{C}}(z_1, z_2) = \|P_1P_2\| \le 1$

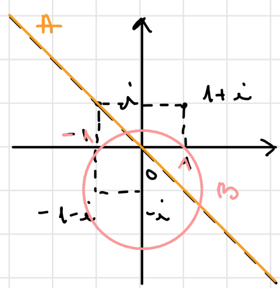
b) $A = \left\{ z \in \mathbb{C},\ \dfrac{|z-(1+i)|}{\underbrace{\cancel{\sqrt{1+|z|^2}}\cdot\sqrt{1+|1+i|^2}}} = \dfrac{|z-(-1-i)|}{\underbrace{\cancel{\sqrt{1+|z|^2}}\sqrt{1+|-1-i|^2}}} \right\}$

$|1+i| = |-1-i| = \sqrt{2}$

$|z-(1+i)| = |z-(-1-i)|$







$$B = \left\{ z \in \mathbb{C},\ \underbrace{\frac{|z-(-i)|}{\sqrt{1+|z|^2}\sqrt{1+|-i|^2}} = \frac{1}{\sqrt{1+|z|^2}}}_{|z-(-i)|=\sqrt{2}} \right\}$$

$$B = \partial U(-i, \sqrt{2})$$

2. Let $(z_n)_{n \in \mathbb{N}^*}$ be in $\mathbb{C}^*$. Let $r_n = |z_n|$, $\theta_n \in \operatorname{Arg} z_n$, $n \in \mathbb{N}^*$. If $r_n \longrightarrow r$, $\theta_n \longrightarrow \theta$, $(n \to \infty)$, then $\lim\limits_{n\to\infty} z_n = r(\cos\theta + i\cdot\sin\theta)$

$$z_n = r_n(\cos\theta_n + i\cdot\sin\theta_n),\ n \in \mathbb{N}^*$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \underset{\downarrow r}{r_n}(\cos \underset{\downarrow \theta}{\theta_n} + i\cdot\sin \underset{\downarrow \theta}{\theta_n}) = r(\cos\theta + i\cdot\sin\theta)$$

3. Find:

a) $\lim\limits_{n\to\infty} \left(\frac{1-i}{3}\right)^n$

$$\boxed{\begin{array}{l} (1)\ \lim\limits_{n\to\infty} z_n = 0 \iff \lim\limits_{n\to\infty} |z_n| = 0 \\ (2)\ \lim\limits_{n\to\infty} z_n = \infty \iff \lim\limits_{n\to\infty} |z_n| = \infty \end{array}}$$

Real Analysis

$$\lim_{n\to\infty} (1+x_n)^{\frac{1}{x_n}} = e \quad (*)$$

$$\lim_{n\to\infty} x_n = 0$$

$e^x = \ldots$

$e^i = ???$

$e^{x+iy} = ???$

$$\left|\left(\frac{1-i}{3}\right)^n\right| = \left|\frac{1-i}{3}\right|^n = \left(\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2}\right)^n = \left(\frac{\sqrt{2}}{3}\right)^n \xrightarrow{(1)} 0$$

$$\frac{\sqrt{2}}{3} \in (0,1)$$

$$\overset{(1)}{\Rightarrow} \lim_{n\to\infty} \left(\frac{1-i}{3}\right)^n = 0$$

b) $\displaystyle\lim_{n\to\infty} \frac{z^n}{1+z^{2n}} = ?$, $z \in \mathbb{C}$, $|z| \neq 1$

<u>Obs.</u>

$z = -1 \Rightarrow \displaystyle\lim_{n\to\infty} \frac{(-1)^n}{2}$ does <u>NOT</u> exist.

- $|z| < 1 \Rightarrow |z^n| = |z|^n \longrightarrow 0 \overset{(1)}{\Longrightarrow} \displaystyle\lim_{n\to\infty} z^n = 0$,

$$\lim_{n\to\infty} z^{2n} = 0 \Rightarrow \lim_{n\to\infty} \frac{z^n}{1+z^{2n}} = \frac{0}{1+0} = 0$$

$\circ \ |z| > 1 \Rightarrow |z^n| \longrightarrow \infty \Rightarrow \left|\frac{1}{z^n}\right| = \left(\frac{1}{|z|}\right)^n \longrightarrow 0 \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} \frac{z^n}{1+z^{2n}} = \lim_{n\to\infty} \frac{\frac{1}{z^n}}{\frac{1}{z^n}+1} = \frac{0}{0+1} = 0$$

$$\Downarrow$$

$$\lim_{n\to\infty} \frac{1}{z^n} = 0$$

$$\lim_{n\to\infty} \frac{1}{z^{2n}} = 0$$

c) $\displaystyle \lim_{n\to\infty} \frac{(1+i\sqrt{3})^n}{n^2} = ?$

$$\circ \ \left|\frac{(1+i\sqrt{3})^n}{n^2}\right| = \frac{|1+i\sqrt{3}|^n}{n^2} = \frac{(\sqrt{1+3})^n}{n^2} = \frac{2^n}{n^2} \longrightarrow \infty \Rightarrow$$

$$\overset{(2)}{\Longrightarrow} \lim \frac{(1+i\sqrt{3})^n}{n^2} = \infty$$

$$\lim_{x\to\infty} \frac{2^x}{x^2} \overset{l'H}{=} \lim_{x\to\infty} \frac{2^x \cdot \ln 2}{2x} = \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} = \infty$$

$$\boxed{\begin{aligned} \lim_{\substack{n\to\infty \\ (x\in\mathbb{R})}} \left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty} \left[\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right]^x \\ &= e^x \end{aligned}}$$

(*)

4. Let $z = x + i\cdot y \in \mathbb{C}$. Prove that:

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^x(\cos y + i\cdot\sin y)$$

Let $z_n = \left(1+\frac{z}{n}\right)^n$, $w_n = 1 + \frac{z}{n}$, $n\in\mathbb{N}^*$

St. 1: $$\lim_{n\to\infty} r_n = \lim_{n\to\infty}\Big(1 + \underbrace{\frac{2x}{n} + \frac{x^2+y^2}{n^2}}_{x_n}\Big)^{\frac{n}{2}} =$$

$$= \lim_{n\to\infty}\left[(1+x_n)^{\frac{1}{x}}\right]^{\frac{n}{2}\cdot x_n} \overset{*}{=} e^{\lim_{n\to\infty}\frac{n}{2}\left(\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)} =$$

$$= e^{x + \lim_{n\to\infty}\frac{x^2+y^2}{2n}} = e^x$$

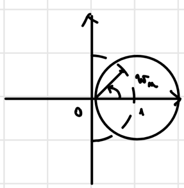
St. 2: Let $\theta_n = n\cdot \arg w_n \in \operatorname{Arg} z_n = \operatorname{Arg} w_n^n =$

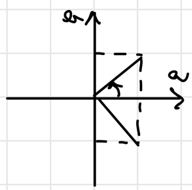
$= \{n\cdot\arg w_n + 2k\pi : k\in\mathbb{Z}\}$

$\downarrow$

De Moivre's formula (seminar 1)

$$\arg w_n = \arg\left(1+\frac{z}{n}\right)$$





- $\arg(a+i\cdot b) = \operatorname{atan}\frac{b}{a}$, $a,b>0$
- $\arg(a-i\cdot b) = \operatorname{atan}\frac{-b}{a}$

Since $1+\frac{z}{n}\to 1$, $n\to\infty$, $\exists\, N\in\mathbb{N}^*$ s.t. $1+\frac{z}{n}$ is the I or IV. quadrant, $\forall\, n\in\mathbb{N}$.

Pb. 2, Seminar 1 $\Rightarrow \arg w_n = \arg\left(1+\frac{x}{n}+i\cdot\frac{y}{n}\right) =$

$= \operatorname{atan} \dfrac{\frac{y}{n}}{1+\frac{x}{n}}, \ \forall n \geq N$

5. a) $\lim\limits_{n\to\infty}\left(1-\frac{i\pi}{2n}\right)^n = ?$

H: a, b

b) $\lim\limits_{n\to\infty}\left(\frac{4ni-\pi}{4ni}\right)^n = ?$

SEMINAR 4

30. X. 2023

1. $$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n \overset{(*)}{=} e^x\left(\cos y + i\cdot \sin y\right),\ \forall z = x + i\cdot y \in \mathbb{C}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x,\ \forall x\in\mathbb{R}$$

ex: $$\lim_{n\to\infty}\left(1+\frac{\pi}{i\cdot n}\right)^n \underset{z=\frac{\pi}{i}}{\overset{(*)}{=}} e^0\left(\underbrace{\cos(-\pi)}_{-1} + i\cdot \underbrace{\sin(-\pi)}_{0}\right) = -1$$

$$= \cancel{\lim_{n\to\infty}\left[\left(1+\frac{\pi}{i\cdot n}\right)^{\frac{i\cdot n}{\pi}}\right]^{\frac{\pi}{i}}}$$

$$z = \frac{i\cdot\pi}{i\cdot i} = -i\pi$$

$$\boxed{\lim_{x\to 0}\frac{tg\, x}{x} = 1}$$

From Seminar 3, pb. 4: ... $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty} n\cdot \arg z_n =$

$$= \lim_{n\to\infty} n\cdot arctg\,\frac{\frac{y}{n}}{1+\frac{x}{n}}$$

$$= \lim_{n\to\infty} \frac{arctg\,\frac{\frac{y}{n}}{1+\frac{x}{n}}}{tg\left(arctg\,\frac{\frac{y}{n}}{1+\frac{x}{n}}\right)}\cdot \frac{\frac{y}{n}}{1+\frac{x}{n}}\cdot n = y$$

Notation:

$$e^z = e^{x+i\cdot y} := \underline{e^x(\cos y + i\cdot\sin y)}, \quad z = x+i\cdot y\in\mathbb{C}$$

1. Solve in $\mathbb{C}$:

a) $e^z = 1$ b) $e^{\bar z} = i$ c) $e^{(1-i)z} = \dfrac{1-i\sqrt{3}}{1-i}$

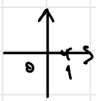
a) $e^z = 1 \iff e^{x+i\cdot y} = \underset{1+i\cdot 0}{\underset{\shortparallel}{1}} \iff e^x(\cos y + i\cdot\sin y) =$

$(z = x+i\cdot y)$

$= 1\cdot(\cos 0 + i\cdot\sin 0)$

$|1| = 1$

$$\iff \begin{cases} e^x = 1 \\ y = 0 \pmod{2\pi} \end{cases} \iff \begin{cases} x = 0 \\ y \in \{0+2k\pi,\ k\in\mathbb{Z}\} \end{cases}$$



<u>ex:</u> $e^0 = e^{2\pi i} = e^{-2\pi i} = 1$

b) $e^{\bar z} = i \iff \underbrace{e^{x-i\cdot y}}_{e^{x+i\cdot(-y)}} = \underset{0+1\cdot y}{i} \iff e^x(\cos y + i\cdot\sin y) =$

$|i| = 1$

$e^{\bar z} = \overline{(e^z)}$

$= 1\left(\cos\frac{\pi}{2} + i\cdot\sin\frac{\pi}{2}\right)$

$$\iff e^x(\cos(-y) + i\cdot\sin(-y)) = 1\left(\cos\left(\tfrac{\pi}{2}\right) + i\cdot\sin\left(\tfrac{\pi}{2}\right)\right)$$

$$\iff \begin{cases} e^x = 1 \\ -y = \frac{\pi}{2} \pmod{2\pi} \end{cases} \iff \begin{cases} x = 0 \\ y \in \{-\frac{\pi}{2} + 2k\pi,\ k\in\mathbb{Z}\} \end{cases}$$

$$\iff z_k = i\left(\frac{\pi}{2} + 2k\pi\right),\ k\in\mathbb{Z}$$

c) $e^{(1-i)z} = \dfrac{1-i\sqrt{3}}{1-i}$

$1-i\sqrt{3} = 2\cdot\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\cdot\sin\left(-\frac{\pi}{3}\right)\right)$

$|1-i\sqrt{3}| = \sqrt{1^2+(-\sqrt{3})^2} = 2$

$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\cdot\left(-\frac{1}{\sqrt{2}}\right)\right) = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\cdot\sin\left(-\frac{\pi}{4}\right)\right)$

$|1-i| = \sqrt{2}$

$\dfrac{1-i\cdot\sqrt{3}}{1-i} = \dfrac{2}{\sqrt{2}}\left(\cos\left(\underbrace{-\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)}_{-\frac{\pi}{12}}\right) + i\cdot\sin\left(\underbrace{-\frac{\pi}{3}-\left(\frac{\pi}{4}\right)}_{-\frac{\pi}{12}}\right)\right)$

$z = x+i\cdot y$

$e^{(1-i)(x+i\cdot y)} = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\cdot\sin\left(-\frac{\pi}{12}\right)\right)$

$\Leftrightarrow e^{x+y+i(y-x)} = e^{x+y}\left(\cos(y-x) + i\sin(y-x)\right) =$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\cdot\sin\left(-\frac{\pi}{12}\right)\right)$

$\Leftrightarrow \begin{cases} e^{x+y} = \sqrt{2} \\ y-x = -\frac{\pi}{12} \pmod{2\pi} \end{cases}$

$\begin{cases} x+y = \ln\sqrt{2} = \frac{1}{2}\ln 2 \\ x-y \in \left\{\frac{\pi}{12} + 2k\pi,\ k\in\mathbb{Z}\right\} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x_k + y_k = \frac{1}{2}\ln 2 \\ x_k - y_k = \frac{\pi}{12} + 2k\pi \end{cases},\ k\in\mathbb{Z}$

$$\Leftrightarrow \begin{cases} x_k = \frac{1}{2}\left(\frac{1}{2}\ln 2 + \frac{\pi}{12} + 2k\pi\right) \\ y_k = \frac{1}{2}\left(\frac{1}{2}\ln 2 - \frac{\pi}{12} - 2k\pi\right) \end{cases}$$

$$\Leftrightarrow \quad z_k = \frac{1}{2}\left(\frac{1}{2}\ln 2 + \frac{\pi}{12} + 2k\pi + i\cdot\left(\frac{1}{2}\ln 2 - \frac{\pi}{12} + 2k\pi\right)\right), \quad k \in \mathbb{Z}$$

2. Establish if the following functions extend continuously at $z_0 = 0$.

a) $f: \mathbb{C}^* \longrightarrow \mathbb{C}$, $f(z) = \dfrac{\operatorname{Im} z}{z}$, $z \in \mathbb{C}^*$

b) $f: \mathbb{C}^* \longrightarrow \mathbb{C}$, $f(z) = \dfrac{z \cdot \operatorname{Im} z}{|z|}$, $z \in \mathbb{C}^*$

If the continuous extension exists, is the extension differentiable at $z_0 = 0$?

$f: \mathbb{C}^* \longrightarrow \mathbb{C}$ has a cont. ext. at $z_0 \Leftrightarrow$

$\Leftrightarrow \exists\, \tilde{f}: \mathbb{C} \longrightarrow \mathbb{C}$ s.t. $\tilde{f}\big|_{\mathbb{C}^*} = f$ and

$\tilde{f}$ is cont. at z_0 $\left(\lim\limits_{z \to z_0} \underbrace{\tilde{f}(z)}_{f(z)} = \tilde{f}(z_0)\right)$

$\Leftrightarrow \exists \lim\limits_{z \to z_0} f(z) \in \mathbb{C}$

a) $? \exists \lim_{z \to 0} \frac{\operatorname{Im} z}{z} = ?$

- $z_n = \frac{1}{n} \to 0, \ n \to \infty$

$$\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} \frac{0}{\frac{1}{n}} = 0$$

- $z_n = \frac{i}{n} \to 0, \ n \to \infty$

$$\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{i}{n}} = \frac{1}{i} \neq 0$$

$\Rightarrow \nexists \lim_{z \to z_0} f(z)$

b) $? \exists \lim_{z \to 0} \frac{z \cdot \operatorname{Im} z}{|z|} = ?$

Ioana + Andra

$$\boxed{\lim_{z \to z_0} f(z) = 0 \iff \lim_{z \to z_0} |f(z)| = 0}$$

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \underbrace{\left| \frac{z \cdot \operatorname{Im} z}{|z|} \right|}_{\frac{|\cancel{z} \operatorname{Im} z|}{\cancel{|z|}}} = \lim_{z \to 0} |\operatorname{Im} z| = 0$$

$\Rightarrow \exists \lim_{z \to 0} f(z) = 0$

So, $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{C}, \ \tilde{f}(z) = \begin{cases} \frac{z \cdot \operatorname{Im} z}{|z|}, & z \in \mathbb{C}^* \\ 0, & z = 0 \end{cases}$

is a cont. extension of f at $z_0 = 0$

f diff. at z_0, if $\exists \underbrace{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$ $\in \mathbb{C}$

$$\lim_{z \to 0} \frac{\frac{z \cdot \operatorname{Im} z}{|z|} - 0}{z - 0} = \lim_{z \to 0} \frac{\operatorname{Im} z}{|z|}$$

- $z_n = \frac{1}{n}$

$$\lim_{n \to 0} \frac{\operatorname{Im} \frac{1}{n}}{\left|\frac{1}{n}\right|} = 0$$

- $z_n = \frac{i}{n}$

$$\lim_{n \to 0} \frac{\operatorname{Im} \frac{i}{n}}{\left|\frac{i}{n}\right|}$$

So, $\nexists \lim_{z \to 0} \frac{\operatorname{Im} z}{|z|}$

3. Find $a, b, c \in \mathbb{C}$ s.t. $f : \mathbb{C} \longrightarrow \mathbb{C}$,

$$f(x + i \cdot y) = (x + a \cdot y) + i \cdot (bx + c \cdot y), \quad x + i \cdot y \in \mathbb{C}$$

is differentiable at any $z = x + i \cdot y$ in $\mathbb{C}$ and find $f'(z)$

$f = u + i \cdot v : D \longrightarrow \mathbb{C}$ is differentiable at z_0 $\Leftrightarrow$

$\Leftrightarrow f$ is $\mathbb{R}$-differentiable at z_0 and

Cauchy-Riemann theorem

$z_0 = x_0 + i \cdot y_0 \in D$

$$\begin{cases} \dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0) \\[2ex] \dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0) \end{cases}$$

If $u, v \in C^1(D)$ (u, v have continuous partial derivatives), then $f = u + i \cdot v$ is $\mathbb{R}$-diff.

$$\begin{aligned} u(x,y) &= \underline{x + a \cdot y} \\ v(x,y) &= \underline{bx + cy} \end{aligned}, \quad x + i \cdot y \in \mathbb{C} \Rightarrow u, v \in C^1(\mathbb{C}) \Rightarrow$$

$\Rightarrow f$ is $\mathbb{R}$-diff.

$$\begin{cases} \dfrac{\partial}{\partial x}(x + a \cdot y) = \dfrac{\partial}{\partial y}(bx + cy) \\[2ex] \dfrac{\partial}{\partial y}(x + a \cdot y) = \dfrac{\partial}{\partial x}(b \cdot x + c \cdot y) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 1 = c \\ a = -b \end{cases}$$

$$f_a(x+i\cdot y) = (x+a\cdot y) + i\cdot(-ax+y), \quad x+i\cdot y \in \mathbb{C}$$

$\Rightarrow f_a$ is diff. on $\mathbb{C}$, $\forall\, a \in \mathbb{C}$

$$f_a'(x+i\cdot y) = ?$$

$$\boxed{f'(z_0) = \frac{\partial f}{\partial x}(z_0)}$$

$$f_a'(x+iy) = \frac{\partial f_a}{\partial x}(x+i\cdot y) = \frac{\partial}{\partial x}\Big((x+a\cdot y) + i(-ax+y)\Big)$$

$$= 1 - a\cdot i, \quad \forall\, x+i\cdot y \in \mathbb{C}$$

SEMINAR 5

6. XI. 2023

H: C

1. Solve in $\mathbb{C}$:

a) $\cos z = -i$

b) $\sin \bar{z} + i \cdot \cos \bar{z} = 1$

c) $\sin z = 2$

$$e^{x+i\cdot y} = e^x(\cos y + i \cdot \sin y)$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

a) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\qquad z = x + i \cdot y$

$$\frac{e^{iz} + e^{-iz}}{2} = -i \iff e^{iz} + e^{-iz} + 2i = 0 \iff$$

$$\iff e^{iz} + \frac{1}{e^{iz}} + 2i = 0$$

$$\boxed{e^z \neq 0, \ \forall z \in \mathbb{C}}$$

<u>Denote:</u> $t = e^{iz} \neq 0$

$$t + \frac{1}{t} + 2i = 0$$

$$t^2 + 2it + 1 = 0$$

$$\Delta = (2i)^2 - 4 \cdot 1 \cdot 1 = -8$$

$$\Rightarrow t_{1,2} = \frac{-2i \pm \sqrt{-8}}{2} = \frac{-2i \pm i \cdot 2\sqrt{2}}{2}$$

$$= (-1 \pm \sqrt{2})i$$

$$\sqrt{-8} = \pm i \sqrt{8} = \pm i\, 2\sqrt{2}$$

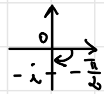
$$|x + i \cdot y| = \sqrt{x^2 + y^2}$$

I. $e^{iz} = (-1-\sqrt{2})\,i \iff e^{i(x+i\cdot y)} = -(1+\sqrt{2})\,i$

$z = x+i\cdot y \qquad \iff e^{-y+i\cdot x} = -(1+\sqrt{2})\,i$

$\iff e^{-y}(\cos x + i\cdot \sin y) = (1+\sqrt{2})(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}))$

$-(1+\sqrt{2})\,i = ?\cdot(\cos ? + i\cdot \sin ?)$



$|-(1+\sqrt{2})\,i| = 1+\sqrt{2}$

$-(1+\sqrt{2})\,i = (1+\sqrt{2})\cdot(\underbrace{\cos(-\frac{\pi}{2}) + i\cdot \sin(-\frac{\pi}{2})}_{-i})$

$$\iff \begin{cases} e^{-y} = 1+\sqrt{2} \\ x = -\frac{\pi}{2} \pmod{2\pi} \end{cases} \iff \begin{cases} y = -\ln(1+\sqrt{2}) \\ x \in \{-\frac{\pi}{2} + 2k\pi, k\in\mathbb{Z} \end{cases}$$

$$\iff z_k = -\frac{\pi}{2} + 2k\pi - i\ln(1+\sqrt{2}),\ k\in\mathbb{Z}$$

II. $e^{iz} = (-1+\sqrt{2})\,i \iff e^{-y}(\cos x + i\cdot \sin x) = (-1+\sqrt{2})(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$z = x+i\cdot y$

$(-1+\sqrt{2})\,i = (-1+\sqrt{2})(\cos\frac{\pi}{2} + i\cdot\sin\frac{\pi}{2})$

$|(-1+\sqrt{2})\,i| = -1+\sqrt{2}$

$$\iff \begin{cases} e^{-y} = \sqrt{2}-1 \\ x = \frac{\pi}{2} \pmod{2\pi} \end{cases} \iff \begin{cases} y = -\ln(\sqrt{2}-1) \\ x \in \{\frac{\pi}{2} + 2k\pi, k\in\mathbb{Z}\} \end{cases}$$

$$\iff z_k = \frac{\pi}{2} + 2k\pi - i\cdot\ln(\sqrt{2}-1),\ k\in\mathbb{Z}$$

b) $\sin \bar{z} + i \cdot \cos \bar{z} = 1$

$$\sin \bar{z} = \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} \qquad \cos \bar{z} = \frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2}$$

$$\frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} + i\,\frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2} = 1 \quad \Big| \cdot 2i$$

$$e^{i\bar{z}} - e^{-i\bar{z}} + i \cdot i\left(e^{i\bar{z}} + e^{-i\bar{z}}\right) = 2i$$

$$\cancel{e^{i\bar{z}}} - e^{-i\bar{z}} - \cancel{e^{i\bar{z}}} - e^{-i\bar{z}} = 2i$$

$$-2e^{-i\bar{z}} = 2i$$

$$e^{-i\bar{z}} = -i \iff e^{y-ix} = -i \iff e^{-y}(\cos(-x) + i \cdot \sin(-x)) =$$

$z = x + iy$

$$= 1\left(\cos\left(-\frac{\pi}{2}\right) + i \cdot \sin\left(-\frac{\pi}{2}\right)\right)$$

$-i \cdot \bar{z} = (-i)(x - iy) = y - ix$

$$\iff \begin{cases} e^{-y} = 1 \\ -x = -\frac{\pi}{2} \pmod{2\pi} \end{cases} \iff$$

$$\iff z_k = \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$$

2. $f : \mathbb{C}^* \longrightarrow \mathbb{C}$, $f(z) = 3z^2 + 5\underbrace{|z|^2}_{z\cdot\bar{z}} - 6\bar{z}^2 + 2z - 7\bar{z} + \frac{1}{z} + 10$, $z \in \mathbb{C}^*$

Find all $z \in \mathbb{C}^*$ s.t. f is diff. at z and $f'(z)$.

$|z|^2 = z \cdot \bar{z}$

$$\boxed{f \text{ is diff at } z_0 \iff f \text{ is } \mathbb{R}\text{-diff at } z_0 \text{ and } \frac{\partial f}{\partial \bar{z}}(z_0) = 0}$$

$$\boxed{f \text{ diff. at } z_0 \Rightarrow f'(z_0) = \frac{\partial f}{\partial z}(z_0)}$$

$$\parallel$$

$$\boxed{\frac{\partial f}{\partial x}(z_0)}$$

$f = u + iv : \Delta \longrightarrow \mathbb{C}$

If $u, v \in C^1(\Delta)$ (cont. partial deriv.),

then f is $\mathbb{R}$-diff on Δ.

Re f, Im f are ratios of pol. functions.

$\Rightarrow$ Re f, Im $f \in C^1(\mathbb{C}^*)$

$\Rightarrow f$ is $\mathbb{R}$-diff. on $\mathbb{C}^*$

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{\partial}{\partial \bar{z}}\left(3z^2 + 5z\cdot\bar{z} - 6\bar{z}^2 + 2z - 7\bar{z} + \frac{1}{z} + 10\right)\Bigg|_{z=z_0}$$

$$= \left(3\cdot 0 + 5z\cdot 1 - 6\cdot 2\cdot \bar{z} + 0 - 7 + 0 + 0\right)\Big|_{z=z_0}$$

$$= 5z_0 - 12\bar{z}_0 - 7 = 0 \quad \Leftrightarrow \quad z_0 = x_0 + i\cdot y_0$$

$$5x_0 + 5\cdot i y_0 - 12x_0 + 12iy_0 - 7 = 0$$

$$\Leftrightarrow \begin{cases} 5x_0 - 12x_0 - 7 = 0 \\ 5y_0 + 12iy_0 = 0 \end{cases} \Rightarrow \begin{cases} x_0 = -1 \\ y_0 = 0 \end{cases} \Leftrightarrow \boxed{z_0 = -1}$$

$$f'(-1) = \frac{\partial f}{\partial z}(-1) = \frac{\partial}{\partial z}\left(3z^2 + 5z\cdot\bar{z} - 6\bar{z}^2 + 2z - 7\bar{z} + \frac{1}{z} + 10\right)\Big|_{z=-1}$$

$$= \left(3\cdot 2z + 5\bar{z}\cdot 1 - 0 + 2 - 0 - \frac{1}{z^2} + 0\right)\Big|_{z=-1}$$

$$= -10$$

$$\left(\frac{1}{z}\right)' = \frac{\overset{=0}{1'}\cdot z - 1\cdot \overset{=1}{z'}}{z^2} = -\frac{1}{z^2}$$

3. Find $A, B, C \in \mathbb{C}$ s.t. $f: \mathbb{C} \to \mathbb{C}$,

$f(z) = A\cdot z^2 + B|z|^2 + C\bar{z}^2$, $z \in \mathbb{C}$ is an entire function (holomorphic on $\mathbb{C}$ = diff. at any $z \in \mathbb{C}$) and f'!

$\underbrace{\text{Re } f, \text{Im } f}_{\in C^1(\mathbb{C})}$ are pol. functions $\Rightarrow f$ is $\mathbb{R}$-diff on $\mathbb{C}$

f is entire $\iff$ $\underline{\underline{\dfrac{\partial f}{\partial \bar{z}}(z) = 0, \ \forall z \in \mathbb{C}}}$

$$\iff \frac{\partial}{\partial \bar{z}}\left(A \cdot z^2 + B \cdot z \cdot \bar{z} + C \cdot \bar{z}^2\right) = 0, \ \forall z \in \mathbb{C}$$

$$\iff A \cdot 0 + B \cdot z \cdot 1 + C \cdot 2 \cdot \bar{z} = 0, \ \forall z \in \mathbb{C}$$

$$\left.\begin{aligned} &z = 1 \Rightarrow B + 2C = 0 \\ &z = i \Rightarrow iB - 2iC = 0 \ \Big| : i \\ &\qquad\qquad B - 2C = 0 \end{aligned}\right\} \Rightarrow B = C = 0$$

So, $f(z) = A \cdot z^2, \ z \in \mathbb{C}$

$$f'(z) = \frac{\partial}{\partial z}(A z^2) = 2Az, \ z \in \mathbb{C}$$

H: 4

4. $f : \mathbb{C}^* \to \mathbb{C}^*, \ f|z| = |z|^2 + \dfrac{1}{|z|^2}, \ z \in \mathbb{C}^*$.

Find $z \in \mathbb{C}^*$ s.t. f is diff. at z and $f'(z)$.

SEMINAR 6

13. XI. 2023

1. Find all $f \in \mathcal{H}(\mathbb{C})$ s.t.:

a) $\operatorname{Re} f(z) = \underbrace{e^{x} \cdot \cos y}_{u(x,y)}$, $\forall z = x \cdot i \cdot y \in \mathbb{C}$

b) $\operatorname{Im} f(z) = e^{x} \cdot \cos y$, $\forall z = x + i \cdot y \in \mathbb{C}$

c) $|f(z)| = e^{x}(x^2+y^2)$, $\forall z = x + i \cdot y \in \mathbb{C}$

$e^{z} = e^{x+i\cdot y} = e^{x}(\cos y + i \cdot \sin y)$

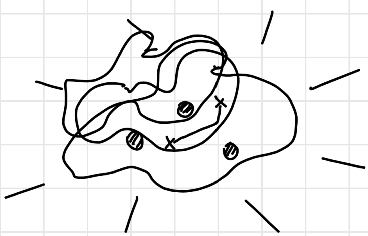
$z = x + i \cdot y$

a) Note that $f(z) = e^{z}$, $z \in \mathbb{C}$, satisfies a)

Theorem 1 (lecture 3)

$D \subseteq \mathbb{C}$ is a domain (open and connected)

$f \in \mathcal{H}(\mathbb{C})$



i) $f \equiv$ const.

$\Updownarrow$

ii) $f' \equiv 0$

iii) $\operatorname{Re} f \equiv c$

$\Updownarrow$

iv) $\operatorname{Im} f \equiv c$

$\Updownarrow$

v) $|f| \equiv$ const

Assume that $g \in \mathcal{H}(\mathbb{C})$ also satisfies a).

Let $h = g - f \in \mathcal{H}(\mathbb{C})$

$$\left.\begin{array}{l} \operatorname{Re} h = 0 \\ \\ \mathbb{C} \text{ dom} \end{array}\right\} \xRightarrow[\text{Lecture}]{T_1} h \equiv c, \text{ where } c \in \mathbb{C} \text{ is a const}$$

$$\Rightarrow g = f + c$$

$\operatorname{Re} h = \operatorname{Re} c = 0 \Rightarrow c = i \cdot \alpha$, where $\alpha \in \mathbb{R}$ is a const.

So, $g(z) = e^z + i \cdot \alpha$, $z \in \mathbb{C}$, satisfies a)

b) Note $f(z) = i \cdot e^z$, $z \in \mathbb{C}$, satisfies b)

$\left(f \in \mathcal{H}(\mathbb{C}),\ \operatorname{Im} f(z) = e^x \cdot \cos y,\ z = x + i \cdot y \in \mathbb{C} \right)$

Assume that $g \in \mathcal{H}(\mathbb{C})$ also satisfies b)

Let $h = g - f \in \mathcal{H}(\mathbb{C})$

$$\left.\begin{array}{l} \operatorname{Im} h \equiv 0 \\ \\ \mathbb{C} \text{ domain} \end{array}\right\} \xRightarrow[\text{Lecture 5}]{\text{Theorem 1}} h \equiv c, \text{ where } c \in \mathbb{C} \text{ const}$$

$\operatorname{Im} h = \operatorname{Im} c = 0 \Rightarrow c = \alpha$,

where $\alpha \in \mathbb{R}$ const.

So, $g(z) = i \cdot e^z + \alpha$, $\forall z \in \mathbb{C}$

c) $|e^z| = |e^x(\cos y + i\cdot\sin y)|$

$$= e^x \underbrace{|\cos y + i\cdot \sin y|}_{i} = e^x$$

$|z^2| = |z|^2 = x^2 + y^2$

Note th $f(z) = e^z \cdot z^2$, $z \in \mathbb{C}$, satisfies c).

Assume $g \in \mathcal{H}(\mathbb{C})$ satisfies c

Let $h(z) = \dfrac{g(z)}{e^z \cdot z^2}$, $z \in \mathbb{C}^*$

$h \in \mathcal{H}(\mathbb{C}^*)$

$$\left.\begin{array}{l} |h(z)| = \dfrac{|g(z)|}{|f(z)|} = 1 \\ \mathbb{C}^1 \text{ domain} \end{array}\right\} \xRightarrow[\text{Lecture 5}]{T_1} h = c, \text{ where } c \in \mathbb{C} \text{ const.}$$

$|h| = |c| = 1$, $c \in \mathbb{U}(0,1)$

$\Rightarrow g(z) = c \cdot e^z \cdot z^2$, $\forall z \in \mathbb{C}^*$

$g \in \mathcal{H}(\mathbb{C}) \Rightarrow g$ cont. at $0 \Rightarrow g(0) = \lim\limits_{z \to 0} \dfrac{g(z)}{c \cdot e^z \cdot z^2} = 0$

So, $g(z) = c \cdot e^z$, $\forall z$ $\quad (c \in \partial\mathbb{U}(0 \quad c \quad)$

d) $\operatorname{Re} f(z) = \underbrace{\cos x \cdot \dfrac{e^y + e^{-y}}{2}}_{u(x,y)}$, $\forall z = x + i\cdot y \in \mathbb{C}$

d) Verify that u is harmonic :

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$$

$$= \frac{\partial}{\partial x}\Bigg(\underbrace{\frac{\partial u}{\partial x}(x,y)}_{-\sin x \cdot \frac{e^{y}+e^{-y}}{2}}\Bigg) + \frac{\partial}{\partial y}\Bigg(\underbrace{\frac{\partial u}{\partial y}(x,y)}_{\cos x \cdot \frac{e^{y}-e^{-y}}{2}}\Bigg)$$

$$= -\cos x \cdot \frac{e^{y}+e^{-y}}{2} + \cos x \cdot \frac{e^{y}+e^{-y}}{2} =$$

$$= 0 \,, \quad \forall\, x+i\cdot y \in \mathbb{C}$$

Let $v = \operatorname{Im} f$

$$f \in \mathcal{H}(\mathbb{C}) \xLeftrightarrow[\text{theorem}]{\text{Cauchy-Riemann}} \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = -\sin x \cdot \frac{e^{} + e^{-y}}{2} \;\Bigg|\; \int_0^y$$

$$\int_0^y \frac{\partial v}{\partial y}(x,t)\,dt = -\sin \cdot \int^y \frac{e^{t}+e^{-t}}{2}\,dt$$

$$\Rightarrow v(x,t)\Big|_0^y = -\sin x \cdot \frac{e^{t}-e^{-t}}{2}\Big|_0$$

$$v(x,y) - v(x,0) = -\sin x \cdot \frac{e^{y} - e^{-y}}{2}, \forall\, x+i\cdot y \in \mathbb{C}$$

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) \qquad x \cdot \frac{e^{y} - e^{-y}}{}$$

$$\frac{\partial v}{\partial x}(x,0) = 0, \forall\, x \in \mathbb{R} \Rightarrow v(x,0) = c, \forall\, x \in \mathbb{R}, \text{ where}$$
$c \in \mathbb{R}$ is a c

$$v(x,y) = \sin x \cdot \frac{e^{-y} - e^{y}}{2} + c, \forall\, x+i\cdot y \in \mathbb{C}$$

So, $f(x+i\cdot y) = u(x,y) + i\cdot v(x,y)$

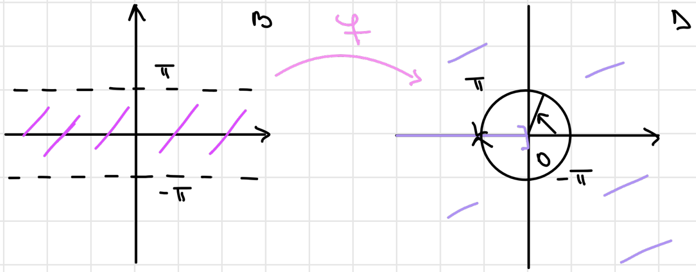
$$= \cos x \cdot \frac{e}{2} + i \cdot \sin \frac{e^{-y} + e^{y}}{2} +$$
$$+ i\cdot c, \forall\, x+i\cdot y \in \mathbb{C}$$

(H): 1.' Write $f(z)$ depending only for f in Problem 1., d).

2. Let $B = \{ z \in \mathbb{C} \quad -\pi < \operatorname{Im} z < \pi \}$

$$D = \mathbb{C} \setminus (-\infty, 0]$$

Prove that $f: B \to D$, $f(z) = e^{z}$, $z \in B$ is well-defined, bijective and $f^{-1}(w) = \ln|w| + i\cdot \arg w$, $w \in D$



proof.

$f(z) \in D, \ \forall z \in B$

$f(x+i\cdot y) = e^x(\cos y + i \sin \ ) \in D, \ \forall \ x \in \mathbb{R}, \ y \in (-\pi, \pi)$

$$\left(z \in D \underset{z \in \mathbb{C}^*}{\Leftrightarrow} z \notin (-\infty, 0] \Leftrightarrow \arg z \neq \pi \right)$$

Recall: $\arg z \in (-\pi, \pi], \ \forall z \in \mathbb{C}^*$

Case: $y = 0 \Rightarrow f(x+i\cdot y) = e^x \in (0, \infty)$

Case: $y \neq 0 \underset{y \in (-\pi, \pi)}{\Rightarrow} \sin y \neq 0 \Rightarrow \operatorname{Im} f(x+i\cdot y) \neq 0 \Rightarrow$

$\Rightarrow f(x+i\cdot y) \notin \mathbb{R}$

$\Rightarrow f(x+i\cdot y) \notin (-\infty, 0]$

We want to prove that the eq: $e^z = w$ has a unique sol. $z \in B$, for every fixed $w \in D$.

$$e^{z} = \underset{\substack{z = x + i\cdot y \\ z \in B}}{} \iff e^{z}(\cos y + i\cdot \sin y) = |w|\,(\cos(\arg w) + i\cdot\sin(\arg w))$$

$$\iff \begin{cases} e^{x} = |w| \\ y = \arg w \pmod{2\pi} \end{cases} \underset{\arg w \in (}{\overset{\iff}{y \in (-\pi, \pi)}} \begin{cases} x = \ln|w| \\ y = \arg w \end{cases}$$

So, f is bij. and $f^{-1}(w) = \ln|w| + i\cdot \arg w$, $w \in \Delta$.

H: 2. $a, b \in \mathbb{R}$, $d_a = \{ a + i\cdot y,\ y \in \mathbb{R} \}$

$d_b = \{ x + i\cdot b,\ x \in \mathbb{R} \}$

Represent graphically d_a, d_b, $\exp(d_a)$, $\exp(d_b)$

SEMINAR 7 20. XI. 2023

Möbius transformations

Def. 1

$f: \mathbb{C}_\infty \longrightarrow \mathbb{C}$, $f(z) = \dfrac{az+b}{cz+d}$, $z \in \mathbb{C}_\infty$, is called a Möbius transformation, where $a, b, c, d \in \mathbb{C}$ s.t. $ad - bc \neq 0$.

Remark 1:

1. If $\boxed{c=0}$, then $\boxed{f(z) = \dfrac{a}{d} z + \dfrac{b}{d}}$, $z \in \mathbb{C}_\infty$, and

$$f(\infty) = \lim_{z \to \infty} f(z) = \infty$$

$$\left(\lim_{z\to\infty} |f(z)| \qquad \lim_{z\to\infty} \left| \underbrace{\frac{a}{d} z + \frac{b}{d}}_{\geq \left|\frac{a}{d}\right| \cdot |z| - \left|\frac{b}{d}\right|} \right| = \infty \right)$$

$$c = 0 \Rightarrow ad - bc = ad \neq 0$$

In this case, f is <u>bijective</u>, $f \in \mathcal{H}(\mathbb{C})$, $f(\infty) = \infty$

2. If $\boxed{c \neq 0}$ then $z_0 = -\frac{d}{c}$ is e the pole of f

and $f(-\frac{d}{c}) = \lim_{z \to -\frac{d}{c}} f(z) = \infty$

In this case, f is bijective, $f \in \mathcal{H}(\mathbb{C} \setminus \{-\frac{d}{c}\}$, $f(-\frac{d}{c}) = \infty$,

$$f(\infty) = \lim_{z \to \infty} f(z) = \lim_{z \to \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} = \frac{a}{c}$$

Remark 2:

If H is the family of Möbius tr ormation, then $(H, \circ)$ is a group, where "$\circ$" is the c sition between functi

H: proof.

Def. 2.

Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ distinct 2 by 2.

The cross-r of z_1, z_2, z_3, z_4 is $(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_1 - z_4} : \frac{z_3 - z_2}{z_3 - z_4}$

$\boxed{P_1}$:

Any Möbius transformation f preserves any cross-ratio.

Proof. :

Let $f(z) = \dfrac{az+b}{cz+d}$, $z \in \mathbb{C}_\infty$. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be

distinct 2 by 2.

We need to pr : $(z_1, z_2, z_3, z_4) = (f(z_1), f(z_2), f(z_3), f(z_4))$

$$(f(z_1), f(z_2), f(z_3), f(z_4)) = \frac{f(z_1) - f(z_2)}{f(z_1) - f(z_4)} : \frac{f(z_3) - f(z_2)}{f(z_3) - f(z_4)} =$$

$= \ldots$ H: com -ete.

$$f(z_j) = \frac{az_j + b}{cz_j + d}, \quad j = \overline{1,4}$$

$\boxed{P_2}$:

Let $z_1, z_2, z_3 \in \mathbb{C}_\infty$ be distinct 2 by 2 and

$w_1, w_2, w_3 \in \mathbb{C}_\infty$ be distinct 2 by 2.

There exists a uniq. Mobius transformation f is the

solution $w = f(z)$.

Proof.

There exists a unique Möbius transformation g is a Möbius transformation s.t.

Let $h = f^{-1} \circ g \in L(G)$, by rem 2. $g(z_j) = w_j$, $j = 1, 2, 3$

$$h(z_j) = f^{-1}(g(z_j)) = f^{-1}(w_j)$$

$$h(z_j) = f^{-1}(w_j) = z_j, j = 1,$$

$$\Leftrightarrow \frac{az_j + b}{cz_j + d} = z_j, \; j = 1, 2, 3$$

$$\Leftrightarrow cz_j^2 + (d-a) z_j - b = 0, \; j = 1, 2, 3 \Leftrightarrow c = d - a = 0 \Leftrightarrow$$

$$\Leftrightarrow h(z) = \frac{a \cdot z + 0}{0 \cdot z + a} = z, \; z \in \mathbb{C}$$

$$\Leftrightarrow h = id_{\mathbb{C}_\infty}$$

$$\Leftrightarrow f^{-1} \circ g = id_{\mathbb{C}_\infty}$$

$$\Leftrightarrow g = f$$

The eq. $cz^2 + (d-a)z - b = 0$ has at m 2 distinct roots, if <u>NOT</u> all its coef. are 0.

lines are circles that pass ∞.

$\boxed{P_3}$:

If $\longrightarrow \mathbb{C}_\infty$, $f(z) = \dfrac{az + b}{cz + d}$, $z \in \mathbb{C}_\infty$, is a

Mobius transformations, then f maps any

<u>generalized circle</u> to a generalized circle.
(circle or line)

- $c = 0 \Rightarrow$ line $\xrightarrow{f}$ line

circle $\xmapsto{f}$ circle

- $c \neq 0 \Rightarrow z_0 = -\frac{d}{c}$ is the <u>pole</u> of f $(f(z_0) = \infty)$

$\in$ line $\xmapsto{f}$ line

$\in$ circle $\xmapsto{f}$ line

$\notin$ line $\xmapsto{f}$ circle

$\notin$ circle $\xmapsto{f}$ circle

<u>proof</u>.

- $c = 0 \Rightarrow$ is clear

- $\neq 0$

$$= \frac{az + b}{cz + d} = \frac{\frac{a}{c} z + \frac{b}{c}}{z + \frac{d}{c}} =$$

$$= \frac{\frac{a}{c} \qquad - \frac{a}{c} \cdot \frac{d}{c} + \frac{b}{c}}{z + \frac{d}{c}} =$$

$$= \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + \frac{d}{c}}$$

$f = f_1 \circ f_2 \circ f_3$

$$f_1(z) = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot z$$

$$f_2(z) = \frac{1}{z}, \quad z \in \mathbb{C}_\infty$$

$$f_3(z) = z + \frac{d}{c}$$

We need only to prove the proposition for f_2.

Hint: use pb. 2, Seminar 2. H: proof.

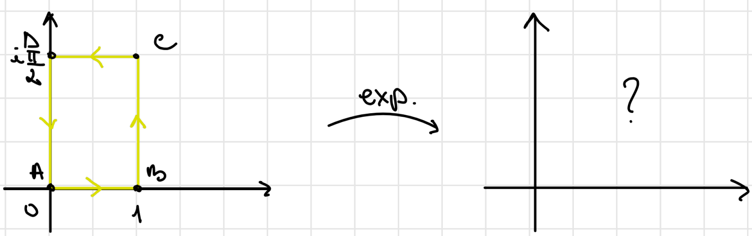
1. $h \quad = ?$

2. Represent graphic $\quad z \in \mathbb{C},\ d_c(z, i) < d_c(z, \infty)\}$

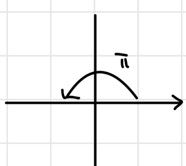
Find all $z \in \mathbb{C}$ s.t. $f'(z) = 0$, where $f : \mathbb{C} \to \mathbb{C}$,

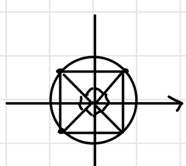
$f(z) = \operatorname{sh} z + i \operatorname{ch} z - iz, \quad z \in \mathbb{C}$

4. Re $\quad$ graphically the image through exp (exponential).



1. $-1 = \cos \pi + \sin \pi$



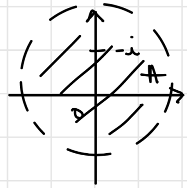


$\sqrt[4]{-1} = \left\{ \cos\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right) + i \sin\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right), \ k = \overline{0,3} \right\}$

$= \left\{ \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \ -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \ -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \ \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right\}$

2. $d_c(z, i) = \dfrac{|z-i|}{\sqrt{1+|z|^2}\sqrt{1+|i|^2}} = \dfrac{|z-i|}{\sqrt{1+|z|^2}\cdot\sqrt{2}}$

$d_c(z, \infty) = \dfrac{1}{\sqrt{1+|z|^2}}$



$z \in A \iff |z-i| < \sqrt{2}$. So, $A = \mathcal{U}(i, \sqrt{2})$

3. $\operatorname{sh} z = \dfrac{e^z - e^{-z}}{2}$ - hyperbolic sinus ; $\operatorname{ch} z = \dfrac{e^z + e^{-z}}{2}$ - hyp. cos.

$\operatorname{sh}' = \operatorname{ch}$, $\operatorname{ch}' = \operatorname{sh}$

$f'(z) = \operatorname{ch} z + i \cdot \operatorname{sh} z$

$\iff t + \dfrac{1}{t} + it - i\dfrac{1}{t} - 2t = 0 \ \Big| \cdot t$

$\iff (1+i) \quad - 2it + (1-i) = 0$

$\Delta = (2i)^2 - 4(1+i)(1-i)$

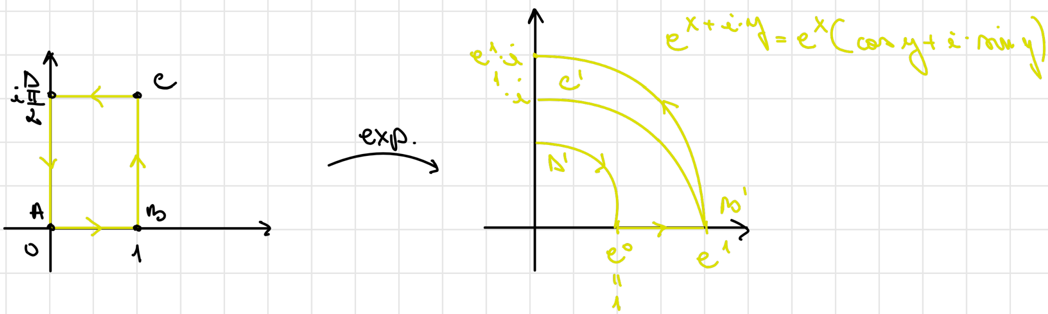
$\Delta = -4 - 8 = -12 = -2^2 \cdot 3$

$t_{1,2} = \dfrac{2i \pm i2\sqrt{3}}{2(} = \dfrac{\overset{1-i}{(1 \pm \sqrt{3})}}{1+i} = \dfrac{1 \pm \sqrt{3}}{2} \cdot (1+i)$

I. $e^{z} = \frac{1+\sqrt{3}}{2}$ ...

II. $e^{z} = \frac{\quad}{2}(-1-i)$ ...

4.



SEMINAR 8

27. XI. 2023

1. a) Find the Möbius transformation $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$ s.t.

$$f(i) = 0, \quad f(-1) = \infty, \quad f(0) = i$$

b) Represent graphically the subset that is contracted by f and the subset that is expanded by f.

a) f preserves the cross-ratio, so we can find $f(z)$ by solving the following eq.: $(z, i, -1, 0) = (w, 0, \infty, i)$, $w = ?$

$$\Leftrightarrow \quad \frac{z-i}{z-0} : \frac{-1-i}{-1-0} = \frac{w-0}{w-i} : \frac{\infty - 0}{\infty - i} \quad \Leftrightarrow *$$

$$\frac{\infty - 0}{\infty - i} = \lim_{f \to \infty} \frac{f-0}{f-i} = \lim_{f \to \infty} \frac{1 - \frac{0}{f}}{1 - \frac{i}{f}} = \frac{1}{1} = 1$$

$$* \quad \frac{z-i}{z} : (1+i) = \frac{w}{w-i} : 1$$

$$wz(1+i) = (w-i)(z-i)$$

$$wz + wzi - wz - iw = -iz + 1$$

$$w(z + zi - z - i) = -iz + 1$$

$$w = \frac{-i(z+i)}{-i(z+1)} \Rightarrow w = \frac{z+i}{z+1}$$

So, $f(z) = \dfrac{-z+i}{z+1}$, $z \in \mathbb{C}_\infty$

Verification, $f(i) = 0$

$$f(-1) = \frac{+1-i}{0} = \infty$$

$$f(0) = \frac{i}{1} = i$$

b) $C = \{z \in \mathbb{C} \setminus \{-1\} ; \ |f'(z)| < 1\} \to$ contracted by f

$E = \{z \in \mathbb{C} \setminus \{-1\} ; \ |f'(z)| > 1\} \to$ expanded by f

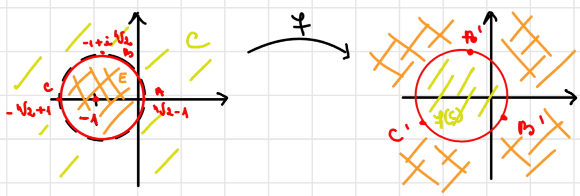
$$f'(z) = \left(\frac{-z+i}{z+1}\right)' = \frac{-(z+1) - (-z+i)\cdot 1}{(z+1)^2} = \frac{-1-i}{(z+1)^2} , \ z \in \mathbb{C} \setminus \{-1\}$$

$$|f'(z)| = \frac{|-1-i|}{|(z+1)^2|} = \frac{\sqrt{2}}{|z+1|^2} , \ z \in \mathbb{C} \setminus \{-1\}$$

$$C = \left\{ z \in \mathbb{C} \setminus \{-1\} \ \middle| \ \frac{\sqrt{2}}{|z+1|^2} < 1 \right\}$$

$$= \{ z \in \mathbb{C} \setminus \{-1\} ; \ |z+1| > \sqrt[4]{2} \} = \mathbb{C} \setminus \overline{U}(-1, \sqrt[4]{2})$$

$$E = \{ z \in \mathbb{C} \setminus \{-1\} \ | \ |z+1| < \sqrt[4]{2} \} = \mathring{U}(-1, \sqrt[4]{2})$$



c) Find $\varphi(C) = ?$, $\varphi(F) = ?$

$$\varphi(z) = \frac{-z+i}{z+1} , \quad z \in \mathbb{C}_\infty$$

$\varphi(\partial U(-1, \sqrt[4]{2}))$

$$\varphi(\sqrt[4]{2} - 1) = \frac{\sqrt[4]{2} + 1 + i}{\sqrt[4]{2} - \cancel{1} + \cancel{1}} = \frac{-\sqrt[4]{2} + 1}{\sqrt[4]{2}} + \frac{i}{\sqrt[4]{2}} \simeq -0.16 + i \cdot 0.84$$

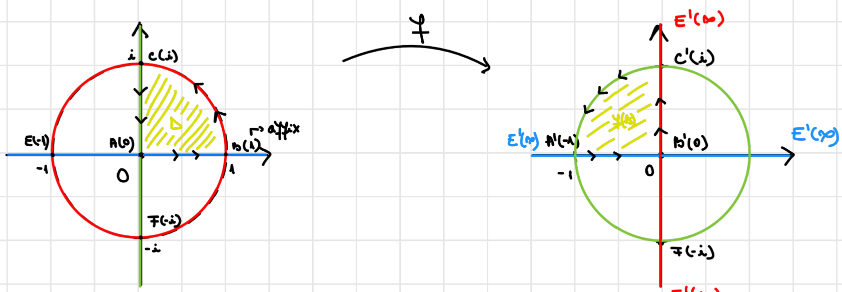
$$\varphi(-\sqrt[4]{2} - 1) = -\frac{\sqrt[4]{2} + 1}{\sqrt[4]{2}} - \frac{i}{\sqrt[4]{2}} \simeq -1.84 - i \cdot 0.84$$

$$\varphi(-1 - i\sqrt[4]{2}) = \frac{1 + (1 - \sqrt[4]{2})i}{i\sqrt[4]{2}} = \frac{\sqrt[4]{2} - 1}{\sqrt[4]{2}} - \frac{i}{\sqrt[4]{2}} \simeq 0.16 - i \cdot 0,84$$

2. $\Delta = \{ z \in \mathbb{C} , \ |z| < 1 , \ \operatorname{Re} z > 0 , \ \operatorname{Im} z > 0 \}$

$$\varphi : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty , \ \varphi(z) = \frac{z-1}{z+1} , \quad z \in \mathbb{C}_\infty$$

$\varphi(\Delta) = ?$



$A \xmapsto{f} A'$	$B \xmapsto{f} B'$	$E \xmapsto{f} E'$
$f(0) = -1$	$f(1) = 0$	$f(-1) = \infty$

- line $A, B, E \xmapsto{f}$ line A', B', E', because it contains ∞

$C \xmapsto{f} C'$

$$f(i) = \frac{i-1}{i+1} = \frac{(i-1)^2}{i^2-1^2} = \frac{-2i}{-2} = i$$

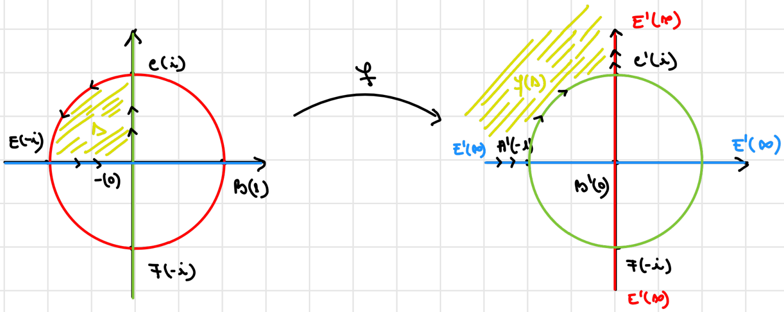
$F \xmapsto{f} F'$

$$f(-i) = \frac{-i-1}{-i+1} = \frac{i+1}{i-1} = \frac{(i+2)^2}{i^2-1^2} = \frac{2i}{-2} = -i$$

- line $C, A, F \xmapsto{f}$ circle $C'A'F'$
- circle $B, C, E \xmapsto{f}$ line B', C', E', because it contains ∞

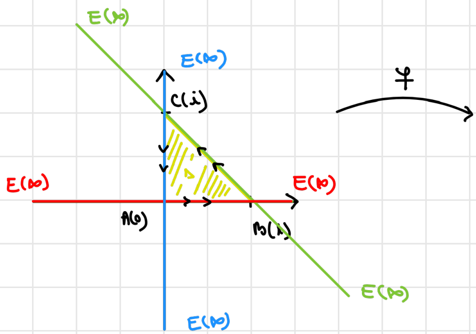
3. $\Delta = \{ z \in \mathbb{C} \,,\, |z| < 1 \,,\, Re\, z < 0 \,,\, Im\, z > 0 \}$

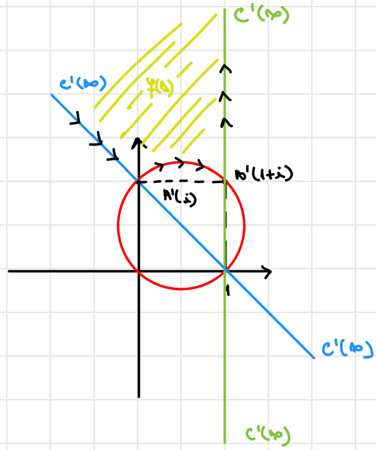
$$f(z) = \frac{z-1}{z+1}, \quad z \in \mathbb{C}, \quad f(\Delta) = ?$$



4. Δ = interior of the triangle with the vertices : $0, 1, i$

$f(z) = \dfrac{z+1}{z-i}$, $z \in \mathbb{C}_\infty$, $f(\Delta) = ?$





$A \xmapsto{f} A'$ $\qquad$ $B \xmapsto{f} B'$

$f(0) = -\dfrac{\frac{i}{1}}{i} = 1$ $\qquad$ $f(1) = \dfrac{\frac{1+i}{2}}{1-i} = \dfrac{2(1+i)}{1^2 - i^2} = 1+i$

$C \xmapsto{f} C'$ $\qquad$ $E \xmapsto{f} E'$

$f(i) = \dfrac{i+1}{0} = \infty$ $\qquad$ $f(\infty) = \dfrac{\infty + 1}{\infty - i} = \lim\limits_{z \to \infty} \dfrac{z+1}{z-i} =$

$= \lim\limits_{z \to \infty} \dfrac{1 + \frac{1}{z}}{1 - \frac{i}{z}} = 1$

5. Find f s.t. f is Möbius and $f(i) = \infty$, $f(-1) = 0$, $f(\infty) = 1$

H: 5.

SEMINAR 9 — 4. XII. 2023

1. a) Find the Möbius transformation f s.t.

$$f(1)=0,\ f(i)=i,\ f(\infty)=1$$

b) Compute the coefficient of deformation ($|f'(z_0)|$) and the angle of rotation ($\arg f'(z_0)$) at $z_0=i$

c) Represent graphically the set in $\mathbb{C}$ that is expanded by f.

a) $(z, 1, i, \infty) = (w, 0, i, 1)$

$$\frac{z-i}{z-\infty} : \frac{i-1}{i-\infty} = \frac{w-0}{w-1} : \frac{i-0}{i-1} \Leftrightarrow$$

$$\Leftrightarrow \frac{z-1}{i-1} \cdot \underbrace{\frac{i-\infty}{z-\infty}}_{1} = \frac{w}{w-1} \cdot \frac{i-1}{i} \Leftrightarrow$$

$$\Leftrightarrow \frac{z-1}{i-1} \cdot \frac{i}{i-1} = \frac{w}{w-1} \Leftrightarrow$$

$$\Leftrightarrow \frac{z-1}{(i-1)^2} \cdot i = \frac{w}{w-1} \Leftrightarrow \frac{z-1}{-2\cancel{i}} \cdot \cancel{i} = \frac{w}{w-1} \Leftrightarrow$$

$$\Leftrightarrow \frac{-2}{z-1} = \underbrace{\frac{w-1}{w}}_{1-\frac{1}{w}} \Big| -1 \Leftrightarrow -\frac{2}{z-1} - \frac{z-1}{z-1} = -\frac{1}{w}$$

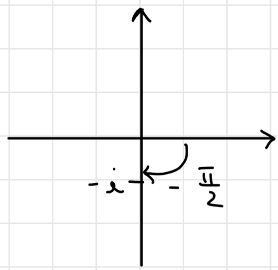
$$\Leftrightarrow \quad \frac{z+1}{z-1} = \frac{1}{w}$$

So, $\varphi(z) = \dfrac{z-1}{z+1}$, $z \in \mathbb{C}_\infty$

b) $$\varphi'(z) = \left(1 - \frac{2}{z+1}\right)' = \frac{2}{(z+1)^2}, \quad z \in \mathbb{C}\setminus\{-1\}$$

$$\varphi'(i) = \frac{2}{(i+1)^2} = \frac{2}{2i} = \frac{1}{i} = -i$$

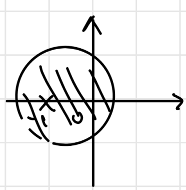
$$|\varphi'(i)| = 1, \quad \underbrace{\arg \varphi'(i)}_{\in(-\pi,\pi)} = \arg(-i) = -\frac{\pi}{2}$$



c) $E = \{z \in \mathbb{C}\setminus\{-1\},\ |\varphi'(z)| > 1\}$

$$E = \left\{z \in \mathbb{C}\setminus\{-1\},\ \frac{2}{|z+1|} > 1\right\}$$

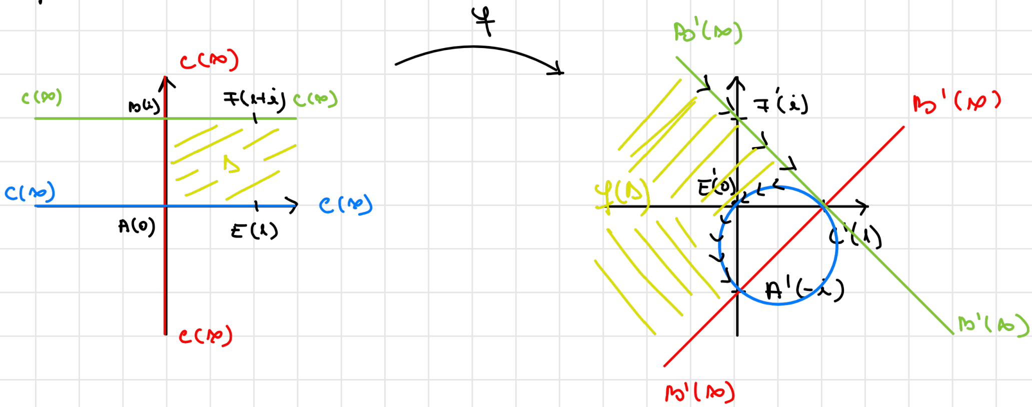
$$E = \{z \in \mathbb{C}\setminus\{-1\},\ |z+1| < \sqrt{2}\} = \dot{U}(-1, \sqrt{2})$$



2. $f(z) = \dfrac{z-1}{z-i}$, $z \in \mathbb{C}_\infty$

$D = \{ z \in \mathbb{C},\ \mathrm{Re}\, z > 0,\ 0 < \mathrm{Im}\, z < 1 \}$ } drawings

$f(D)$



$A \overset{f}{\longmapsto} A'$

$f(0) = \dfrac{1}{i} = -i$

$f(i) = \infty \qquad B \longmapsto B'$

$f(\infty) = 1 \qquad C \longmapsto C'$

<u>line</u> $A, B, C \longmapsto$ <u>line</u> A', B', C'
$\downarrow$
∞

$f(1) = 0$, $E \longmapsto E'$

<u>line</u> $A, E, C \longmapsto$ <u>circle</u> A', E', C'

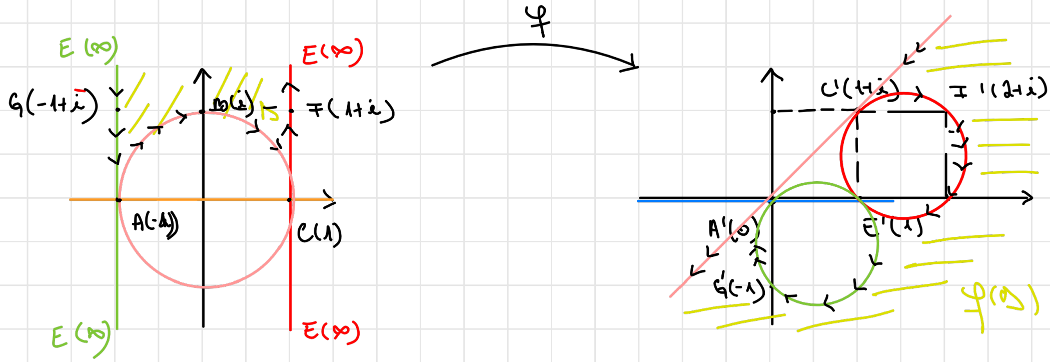
$f(1+i) = \dfrac{i}{1} = i$, $F \longmapsto F'$

line $B, F, C \longmapsto$ line B', F', C'
$\downarrow$
∞

3. $\varphi(z) = \dfrac{z+1}{z-i}$, $z \in \mathbb{C}_\infty$

$D = \{ z \in \mathbb{C} , -1 < \operatorname{Re} z < 1 , \operatorname{Im} z > 0 , |z| > 1 \}$

$\varphi(D)$



$\varphi(-1) = 0$

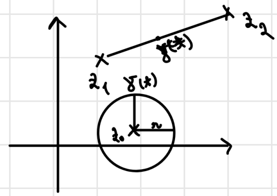
$\varphi(-i) = \infty$

$\varphi(i)$

SEMINAR 10

11. IV. 2023

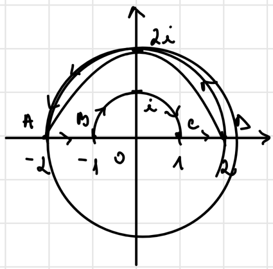
1. Choose a path γ and then compute $\int_\gamma \bar{z}\, dz$ s.t.:



a) $\{\gamma\} = [z_1, z_2]$, $z_1, z_2 \in \mathbb{C}$

b) $\{\gamma\} = \partial U(z_0, r)$, $z_0 \in \mathbb{C}$, $r > 0$

c) $\{\gamma\}$ is as in the fig.



a) $\gamma : [0,1] \longrightarrow \mathbb{C}$

$$\gamma(t) = t \cdot z_2 + (1-t) z_1, \quad t \in [0,1]$$

$$\{\gamma\} = \gamma([0,1]) = [z_1, z_2]$$

$$\int_\gamma \bar{z}\, dz = \int_0^1 \overline{\gamma(t)} \cdot \gamma'(t)\, dt =$$

$$= \int_0^1 (t \cdot \bar{z}_2 + (1-t) \cdot \bar{z}_1) \cdot (z_2 - z_1)\, dt$$

$$= \int_0^1 (\bar{z}_1 + t(\bar{z}_2 - \bar{z}_1))(z_2 - z_1)\, dt$$

$$= \bar{z}_1 (z_2 - z_1) + (\bar{z}_2 - \bar{z}_1)(z_2 - z_1) \cdot \underbrace{\int_0^1 t\, dt}_{\frac{t^2}{2}\Big|_0^1 = \frac{1}{2}}$$

$$= (z_2 - z_1)\left(z_1 + \frac{\bar{z}_2 - \bar{z}_1}{2}\right)$$

$$= \frac{(z_2 - z_1)(\bar{z}_1 + \bar{z}_2)}{2}$$

b) $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$

$\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$

$\{\gamma\} = \partial U(z_0, r)$

$$\int_\gamma \bar{z}\, dz = \int_0^{2\pi} \bar{z}_0 + r \cdot \underbrace{(\overline{e^{it}})}_{e^{-it}} \cdot (0 + r \cdot e^{it} \cdot \underbrace{(it)'}_{=i})\, dt$$

$$= \bar{z}_0 \cdot r \cdot i \int_0^{2\pi} e^{it}\, dt + r^2 i \int_0^{2\pi} \underbrace{e^{-it} \cdot e^{it}}_{e^0 = 1}\, dt$$

$$= \bar{z}_0 \cdot r \cdot i \cdot \underbrace{\frac{e^{it}}{i}\Bigg|_0^{2\pi}}_{=0} + 2\pi r^2 i$$

$e^{2\pi i} = \cos(2\pi) + i \cdot \sin(2\pi)$

$= \cos 0 + i \cdot \sin 0 = 1$

c) $\gamma_1 : [0,1] \longrightarrow \mathbb{C}$

$$\gamma_1(t) = (1-t)\cdot A + tB$$

$$= (1-t)(-2) + t\cdot(-1)$$

$\gamma_2 : [1,2] \longrightarrow \mathbb{C}$

$$\gamma_2(t) = e^{-it\pi}, \quad t \in [1,2]$$

$\gamma_3 : [2,3] \longrightarrow \mathbb{C}$

$$\gamma_3(t) = (3-t)C + (t-2)D$$

γ $$= (3-t)\cdot 1 + (t-2)D, \quad t \in [2,3]$$

$\gamma_4 : [3,4] \longrightarrow \mathbb{C}$

$$\gamma_4(t) = 2e^{i(t-1)\pi}, \quad t \in [3,4]$$

$\gamma : [0,4] \longrightarrow \mathbb{C}$

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$$

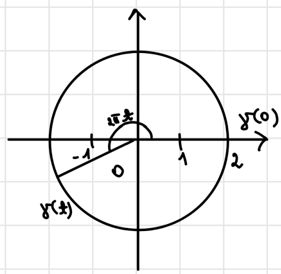
$$\int_\gamma \bar{z} = \int_{\gamma_1} \bar{z} + \int_{\gamma_2} \bar{z} + \int_{\gamma_3} \bar{z} + \int_{\gamma_4} \bar{z}$$

$$\overset{a)}{=} \frac{(B-A)(\bar{B}+\bar{A})}{2} + \frac{(D-C)(\bar{D}+\bar{C})}{2} + \int_1^2 \underbrace{\cancel{e^{it\pi}} \cdot \cancel{e^{-it\pi}}(-i\pi)}_{-i\pi(2-1)}\, dt +$$

$$+ \int_3^4 \underbrace{2\cancel{e^{-i(t-1)\pi}} \cdot (2\cdot \cancel{e^{i(t-1)\pi}} \cdot i\pi)}_{4i\pi}\, dt$$

2. $\displaystyle\int_\gamma \underbrace{\frac{\sin(z^3-z)}{z^3-z}}_{f(z),\ z\in\mathbb{C}\setminus\{0,-1,1\}} dz = ?$, where $\gamma:[0,1]\longrightarrow\mathbb{C}$

$\gamma(t) = 2e^{2\pi i t}$, $t\in[0,1]$



$\underbrace{z^3-z}_{z(z^2-1)} = 0 \iff z\in\{0,-1,1\}$

$f\in\mathcal{H}(\mathbb{C}\setminus\{0,-1,1\}$

$\displaystyle\int_\gamma f = 0$

We use T_3, lecture 9 $\longrightarrow$ $\mathbb{C}$ - starlike w.r.t. 0 $\Rightarrow \displaystyle\int_\gamma f = 0$

? f continuous on $\mathbb{C}$

$\displaystyle \zeta = z^3 - z = \lim \frac{\sin(\zeta)-0}{\zeta-0}$

$$\lim_{z\to z_j}\frac{\sin(z^3-z)}{z^3-z} = \lim_{z\to z_j}\frac{\sin(z^3-z)-\sin(0)}{(z^3-z)-0} = \sin' 0 =$$

$= \cos 0 = 1$,

$z_j\in\{-1,0,1\}$

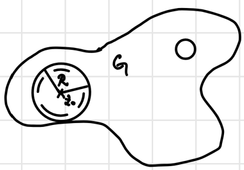
$$f(z) = \begin{cases} \dfrac{\sin(z^3-z)}{z^3-z}, & z\in\mathbb{C}\setminus\{-1,0,1\} \\ 1, & z\in\{-1,0,1\} \end{cases}$$

3. (Mean value property for harmonic functions)

Let $G \subseteq \mathbb{C}$ be open and $u : G \to \mathbb{R}$ be harmonic.

Then, $u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt$, $\forall z_0 \in G, r > 0$

s.t. $\overline{U}(0, r) \subset G$.



Proof.

$\exists R > r$ s.t. $U(z_0, R) \subset G$.

Lecture 8, T_1 $\Rightarrow \exists f \in \mathcal{H}(U(z_0, R))$

s.t. $\operatorname{Re} f = u$ on $U(z_0, R)$

Cauchy's integral formula, Lecture 9

$$\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{\cancel{z_0} + \cancel{re^{it}} - \cancel{z_0}} \cdot \cancel{r \cdot e^{it}} \cdot \cancel{i}\, dt.$$

$(\gamma(t) = z_0 + re^{it},\ t \in [0, 2\pi])$

$$\underbrace{\operatorname{Re} f(z_0)}_{u(z_0)} = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\operatorname{Re} f}_{u}(z_0 + re^{it})\, dt$$

SEMINAR 11 18. XII. 2023

1. Find the power series expansion around z_0 and the radius of convergence for the following functions:

a) $\underset{(1)}{f(z)} = \dfrac{z}{1+z}, \quad z \in \mathbb{C} \setminus \{-1\}; \; z_0 = 0$

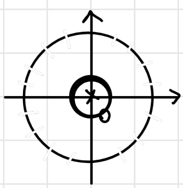
$$f(z) = \sum_{n=0}^{\infty} a_n \cdot \underline{(z-z_0)}^n, \quad z \in \mathcal{U}(0,R)$$

$\hookrightarrow$ power series expansion around z_0

$$\frac{1}{1-y} = 1 + y + y^2 + \dots + y^n + \dots, \quad y \in \mathcal{U}(0,1) \longrightarrow$$

around $y_0 = 0$

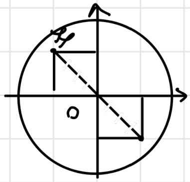
The radius of convergence is $R \in [0, \infty]$ is the largest value s.t. f is holomorphic on $\mathcal{U}(z_0, R)$



So, for (1) $\mathcal{U}(0,1)$ is the largest where f is holomorphic $\Rightarrow R$ = radius of conv. $= 1$

$\forall y \in \mathcal{U}(0,1)$

$y = -z \Rightarrow z = -y \in \mathcal{U}(0,1)$



$$\frac{1}{1-w} = \frac{1}{1+z} = 1 - z + (-z)^2 + \dots + (-z)^n + \dots, \quad z \in \mathbb{U}(0,1)$$

$$\Rightarrow f(z) = \frac{z}{1+z} = z - z^2 + z^3 + \dots + (-1)^{n-1} z^n + \dots,$$

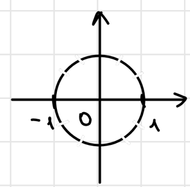
$$z \in \underset{\overset{\parallel}{z_0}}{\underline{\mathbb{U}(0,1)}}$$

b) $f(z) = \dfrac{1}{1-z^2}$, $z \in \mathbb{C} \setminus \{-1, 1\}$; $z_0 = 0$

$$\frac{1}{1-w} = 1 + w + w^2 + \dots + w^n + \dots, \quad w \in \mathbb{U}(0,1)$$

$f \in \mathcal{H}(\mathbb{U}(0,1))$

$R = 1$ = radius of conv.



I. $$\frac{1}{1-z^2} = \frac{(1-z)+(1+z)}{(1-z)(1+z)} \cdot \frac{1}{2} = \frac{1}{2}\left(\frac{1}{1+z} + \frac{1}{1-z}\right) =$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} (-z)^n + \sum_{n=0}^{\infty} z^n\right) = \frac{1}{2}\left(2 + 2z^2 + \dots + 2z^{2n} + \dots\right) =$$

$$= 1 + z^2 + \dots + z^{2n} + \dots, \quad z \in \mathbb{U}(0,1)$$

II. $w = z^2 \Rightarrow f(z) = 1 + z^2 + z^4 + \dots + z^{2n} + \dots, \quad z \in \mathbb{U}(0,1)$

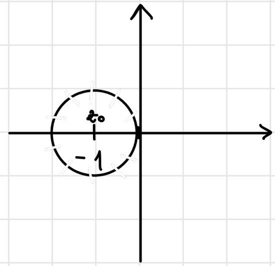
$(z \in \mathbb{U}(0,1) \Rightarrow z^2 \in \mathbb{U}(0,1))$

c) $f(z) = \dfrac{1}{z}, \ z \in \mathbb{C}^* ; \ z_0 = -1$

$f \in \mathcal{U}(-1,1)$

$R=1$ is the largest radius s.t. f is holomorphic on the disk around $z_0 = -1$

$$f(z) = \sum_{n=0}^{\infty} a_n \underbrace{(z-(-1))^n}_{(z+1)^n}$$



$$\frac{1}{1-y} = 1 + y + y^2 + \dots + y^n + \dots, \ y \in \mathcal{U}(0,1)$$

$y = z+1 \qquad z = y-1$

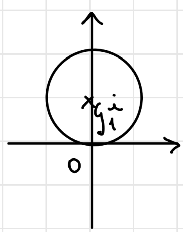
$y \in \mathcal{U}(0,1) \iff z \in \mathcal{U}(-1,1)$

$$\Rightarrow \underbrace{\frac{1}{1-(z+1)}}_{= \frac{1}{z}} = 1 + (z+1) + (z+1)^2 + \dots + (z+1)^n + \dots,$$

$z \in \mathcal{U}(-1,1)$

$$f(z) = -1 - (z+1) - (z+1)^2 - \dots - (z+1)^n - \dots, \ z \in \mathcal{U}(-1,1)$$

d) $f(z) = \dfrac{1}{z}, \ z \in \mathbb{C}^* ; \ z_0 = i$



$$f(z) = \sum_{n=0}^{\infty} a_n (z-i)^n, \ z \in \mathcal{U}(i, R)$$

$f \in \mathcal{H}(\mathcal{U}(i,1))$, $R=1$ is the largest rad.

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots + y^n + \cdots, \quad y \in \mathcal{U}(0,1)$$

$$\frac{1}{z} \overset{z-i+1}{=} \frac{1}{i + (z-i)} \overset{z = y+i-1}{=} \frac{1}{i} \cdot \frac{1}{1 + \frac{z-i}{i}} \overset{y = \frac{z-i}{i}}{=\!=}$$

$$\frac{1}{1-z+i-1} = 1 + (z\ i+1)$$

$$= \frac{1}{i} \cdot \left(1 - \frac{z-i}{i} + \left(\frac{z-i}{i}\right)^2 + \cdots + \left(\frac{z-i}{i}\right)^n + \cdots, \quad z \in \mathcal{U}(i,1)\right.$$

$y \in \mathcal{U}(0,1)$, $z \in \mathcal{U}(i,1)$

$$y = \frac{z-i}{i} \iff z = iy + i$$

$$z = iy + i \in \mathcal{U}(i,1) \iff |iy + i - i| < 1 \iff |i \cdot y| < 1$$

$$\iff |y| < 1$$

So, $$f(z) = \frac{1}{i} - \frac{z-i}{i^2} + \frac{(z-i)^2}{i^3} - \cdots + \frac{(-1)^n}{i^{n+1}} \cdot (z-i)^n + \cdots, \quad z \in \mathcal{U}(i,1)$$

e) $$f(z) = \frac{1}{(1+z)^2}, \quad z \in \mathbb{C} \setminus \{-1\}, \quad z_0 = 0$$

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n, \quad z \in \mathcal{U}(0,1)$$

$$\frac{1}{1-y} = 1 + y + \cdots + y^n + \cdots, \quad y \in \mathcal{U}(0,1)$$

$$(\ )' \Big| \ \frac{1}{1+z} = 1 - z + \cdots + (-1)^{n-1} z^n + \cdots, \quad z \in \mathcal{U}(0,1)$$

Cauchy, Mobius, Riemann

$$-\frac{1}{(1+z)^2} = 0 - 1 + 2z - 3z^2 + \dots + n(-1)^{n-1} z^{n-1} + \dots, \; z \in \mathcal{U}(0,1)$$

(by the Cauchy - Hadamard theorem, Lecture 10, we can differentiate each term)

$$f(z) = 1 - 2z + 3z^2 - \dots + n(-1)^n z^{n-1} + \dots, \; z \in \mathcal{U}(0,1)$$

f) $f(z) = \dfrac{1}{(1-z)^3}, \; z \in \mathbb{C} \setminus \{1\}; \; z_0 = 0$

$$(\;)' \Big| \; \frac{1}{1-z} = 1 + z + z^2 + \dots + z^n + \dots, \; z \in \mathcal{U}(0,1)$$

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \dots + nz^{n-1} + \dots$$

$$\frac{2}{(1-z)^3} = 2 + 6z + \dots + n(n-1)z^{n-2} + \dots, \; z \in \mathcal{U}(0,1)$$

$$f(z) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} \cdot z^{n-2}, \; z \in \mathcal{U}(0,1)$$

$$= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} \cdot z^n$$

g) $f(z) = \dfrac{1}{z^2}, \; z \in \mathbb{C}^*; \; z_0 = i$

$$(\;)' \Big| \; \frac{1}{z} = \frac{1}{i} - \frac{(z-i)}{i^2} + \frac{(z-i)^2}{i^3} - \dots + (-1)^{n+1} \frac{(z-i)^n}{i^{n+1}} + \dots,$$

$$z \in \mathcal{U}(i,1)$$

$$f(z) = \frac{1}{i^2} - \frac{2(z-i)}{i^3} + \dots + \frac{n(-1)^n}{i^{n+1}}(z-i)^n + \dots, \ z \in U(i,1)$$

h) $f(z) = (\cos z)(z+1)$, $z \in \mathbb{C}$; $z_0 = 0$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^n \frac{z^{2n}}{(2n)!} + \dots, \ z \in \mathbb{C}$$

$$f(z) = (\cos z)(z+1) = z - \frac{z^3}{2!} + \frac{z^5}{4!} - \dots + (-1)^n \cdot \frac{z^{2n+1}}{(2n)!} + \dots$$

$$+ 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^n \cdot \frac{z^{2n}}{(2n)!} + \dots$$

$$= 1 + z - \frac{z^2}{2!} - \frac{z^3}{2!} + \frac{z^4}{4!} + \frac{z^5}{4!} - \dots +$$

$$+ (-1)^n \cdot \frac{z^{2n}}{(2n)!} + (-1)^n \frac{z^{2n-1}}{(2n)!} + \dots,$$

$z \in \mathbb{C}$

2. $\lim\limits_{z \to 0} \dfrac{\sin z - z \cdot \cos z}{z^3} = ?$ (with power series)

R: 2

LECTURE 2

10. X. 2023

Topology of the complex plane ($\mathbb{C}$)

$A \subseteq \mathbb{C}$ is :

- open, if $\forall z \in A$ is an interior point $\Leftrightarrow A = \text{int}\, A$ (interior of A)
- closed, if every closure point of A is in $A \Leftrightarrow$
  $\Leftrightarrow A = \text{cl}\, A = \bar{A}$ (closure of A)

Notations :

A' = is the set of accumulation points

∂A = the boundary of A

Properties :

- $A = \bar{A} \Leftrightarrow \mathbb{C} \setminus A$ is open
- $\bar{A} = A \cup A'$
- $\partial A = \bar{A} \setminus \text{int}\, A$

! $A \subseteq \mathbb{C}$ is compact $\Leftrightarrow A$ is closed and bounded.

Def.

A sequence $(z_n)_{n\in\mathbb{N}^*}$ in $\mathbb{C}$ is convergent to $z_0\in\mathbb{C}$, if $\lim\limits_{n\to\infty} |z_n - z_0| = 0$.

Notation:

$z_n \longrightarrow z_0$ or $\lim\limits_{n\to\infty} z_n = z_0$.

Remark:

If $z_n = x_n + iy_n$, $n\in\mathbb{N}^*$ and $z_0 = x_0 + iy_0$, then:

$$\lim_{n\to\infty} z_n = z_0 \iff \lim_{n\to\infty} x_n = x_0 \text{ and } \lim_{n\to\infty} y_n = y_0$$

Propositions:

Let $A \subseteq \mathbb{C}$, $A \neq \emptyset$. Then:

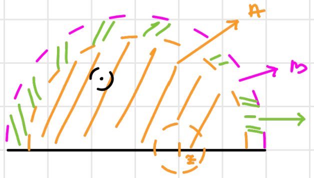
- $z_0 \in \overline{A} \iff \exists (z_n)_{n\in\mathbb{N}^*}$ in A s.t. $\lim\limits_{n\to\infty} z_n = z_0$.
- $z_0 \in A' \iff \exists (z_n)_{n\in\mathbb{N}^*}$ in $A\setminus\{z_0\}$ s.t. $\lim\limits_{n\to\infty} z_n = z_0$
- A is compact $\iff \forall (z_n)_{n\in\mathbb{N}^*}$ in A, $\exists (z_{n_k})_{k\in\mathbb{N}^*}$ in A and $\exists z_0 \in A$ s.t. $\lim\limits_{k\to\infty} z_{n_k} = z_0$

The induced topology

Let $A \subseteq B \subseteq \mathbb{C}$.

Def.

A is open in B, if $\forall z \in A$, $\exists r > 0$ s.t. $U(z,r) \cap B = A$.



B is NOT open in C, but A is open in B

Remark:

A is closed in A with $\lim_{n \to \infty} z_n = z_0 \in B$, we have $z_0 \in A$.

Connected sets in $\mathbb{C}$:

Let $B \in \mathbb{C}$, $B \neq \emptyset$

Def:

B is connected, if the following holds:

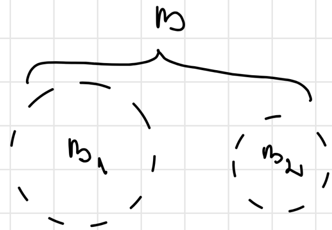
if $A \subseteq B$ is open and closed in B, then either $A = B$ or $A = \emptyset$.

Remark:

D is <u>NOT</u> connected $\iff \exists\, D_1, D_2 \subseteq D$ not empty and <u>open in D</u> s.t. $D_1 \cap D_2 = \emptyset$.

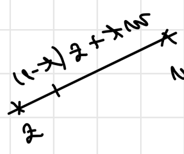
Example:

If D is the reunion of two disjoint open disks, then D is <u>NOT</u> connected.



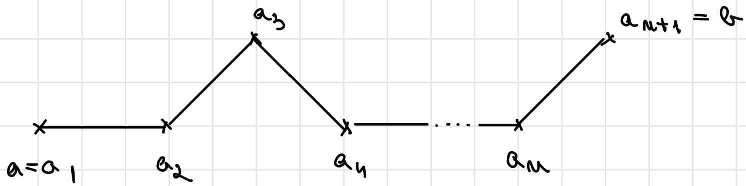
Notation:

for $z, w \in \mathbb{C}$, $[z, w] = \{(1-t)z + tw,\ t \in [0,1]\}$ is the segment z and w.



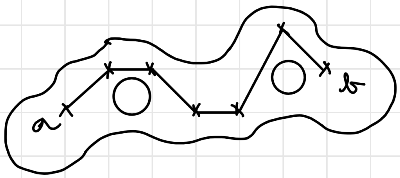
Def:

A <u>polygon</u> from $a \in \mathbb{C}$ to $b \in \mathbb{C}$ is a set of the form $\bigcup_{k=1}^{n} [a_k, a_{k+1}]$, where $a_1 = a, a_2, \dots, a_n \in \mathbb{C}$, $a_{n+1} = b$, $n \in \mathbb{N}^*$



Def.

$A \subseteq \mathbb{C}$ is <u>polygonally connected</u>, if $\forall\, a, b \in A$; $\exists$ a polygon in A from a to b.



Propositions:

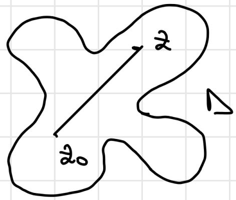
Let $\Delta \subseteq \mathbb{C}$ be <u>open</u>. Then, Δ is <u>connected</u> $\iff$ Δ is <u>polygonally connected</u>.

Def.

$\Delta \subseteq C$ is a <u>domain</u>, if Δ is <u>open</u> and <u>connected</u>.

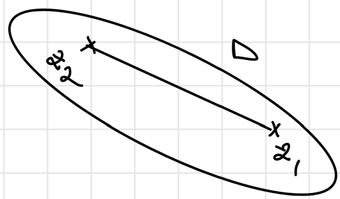
Def.

$\Delta \subseteq \mathbb{C}$ is a <u>starlike</u> domain w.r.t. $z_0 \in \Delta$, if $\forall z \in \Delta$, $[z_0, z] \subset \Delta$.



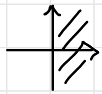
Def.

$\Delta \subseteq \mathbb{C}$ is a <u>convex</u> domain, if $\forall z_1, z_2 \in \Delta$, $[z_1, z_2] \subset \Delta$.

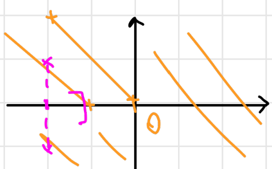


Examples:

- $U(z_0, r)$ is convex in $\mathbb{C}$, $\forall z_0 \in \mathbb{C}$, $r > 0$
- $\{z \in \mathbb{C}, \operatorname{Re} z > 0\}$ is convex (the right-half plane)



- $\mathbb{C} \setminus (-\infty, -1]$ is starlike w.r.t. 0, but it is NOT convex



The stereographic projection:

We shall consider the one point compactification of $\mathbb{C}$,

by adjoining a point, denoted by ∞, which is NOT in $\mathbb{C}$,

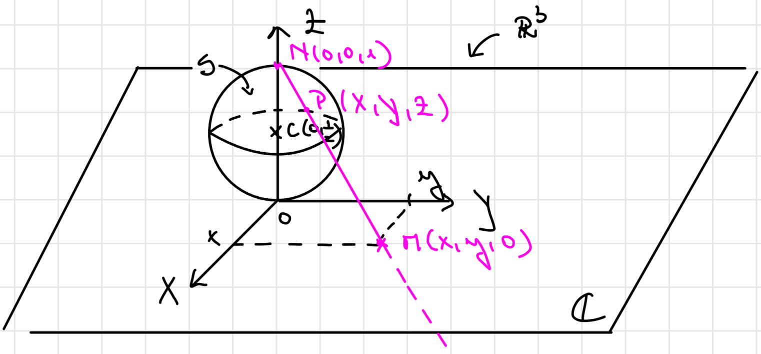
s.t. $\mathbb{C}_\infty := \mathbb{C} \sqcup \{\infty\}$ is a compact space.

↳ is called the extended complex plane

Let $S \subseteq \mathbb{R}^3$ be the sphere $(S): (X-0)^2+(Y-0)^2+(Z-\frac{1}{2})^2 = \frac{1}{4}$

and identify the complex plane with the plane XOY.

S is called the Riemann sphere.



There is a one-to-one correspondence between the points

in $\mathbb{C}$ and the points in $S \setminus \{N\}$:

if $z = x + i \cdot y \in \mathbb{C}$ is the affix of $M(x,y,0)$, then the line

NM intersects $S \setminus \{N\}$ at one point $P(X,Y,Z)$

So, $\varphi: \mathbb{C} \to S\setminus\{N\}$, given by $\varphi(z) = (X, Y, Z)$, $z = x + iy \in \mathbb{C}$, where z is the affix of the stereographic projection of $P(X, Y, Z) \in S\setminus\{N\}$, is <u>bijective</u> and is called the stereographic projection of $\mathbb{C}$.

<u>Proposition</u>:

$$\varphi: \mathbb{C} \to S\setminus\{N\},\ \varphi(z) = \Big(\underbrace{\frac{\operatorname{Re} z}{1+|z|^2}}_{X}, \underbrace{\frac{\operatorname{Im} z}{1+|z|^2}}_{Y}, \underbrace{\frac{|z|^2}{1+|z|^2}}_{Z}\Big),\ z \in \mathbb{C}$$

$$\varphi^{-1}(X, Y, Z) = \underbrace{\frac{X+iY}{1-Z}}_{z},\ (X, Y, Z) \in S\setminus\{N\}$$

We consider the stereographic projection of the north pole to be <u>the point at infinity</u>: $N(0,0,1) \longmapsto \infty \notin \mathbb{C}$

and thus we obtain the <u>extended complex plane</u>.

$$\mathbb{C}_\infty = \mathbb{C} \sqcup \{\infty\}$$

Then $\tilde{\varphi}: \mathbb{C}_\infty \longrightarrow S$, $\tilde{\varphi}(z) = \begin{cases} \varphi(z), & z \in \mathbb{C} \\ (0,0,1), & z = \infty \end{cases}$

is a bijective function called the <u>stereographic projection</u> of the extended complex plane.

LECTURE 3 11. X. 2023

The topology of the extended complex plane:

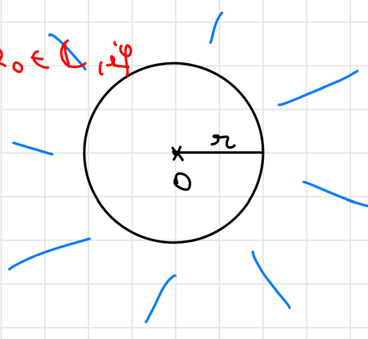
$(S): X^2+Y^2+(Z-\frac{1}{2})^2=\frac{1}{4}$ – the stereographic proj. of $\mathbb{C}_\infty=\mathbb{C}\sqcup\{\infty\}$:
$S\ni N(0,0,1)\longmapsto\infty$

$\tilde{\varphi}:\mathbb{C}_\infty\longrightarrow S$

$$\tilde{\varphi}(z)=\begin{cases}\varphi(z)=\left(\dfrac{\operatorname{Re} z}{1+|z|^2},\dfrac{\operatorname{Im} z}{1+|z|^2},\dfrac{|z|^2}{1+|z|^2}\right), & z\in\mathbb{C}\\ (0,0,1), & z=\infty\end{cases}$$

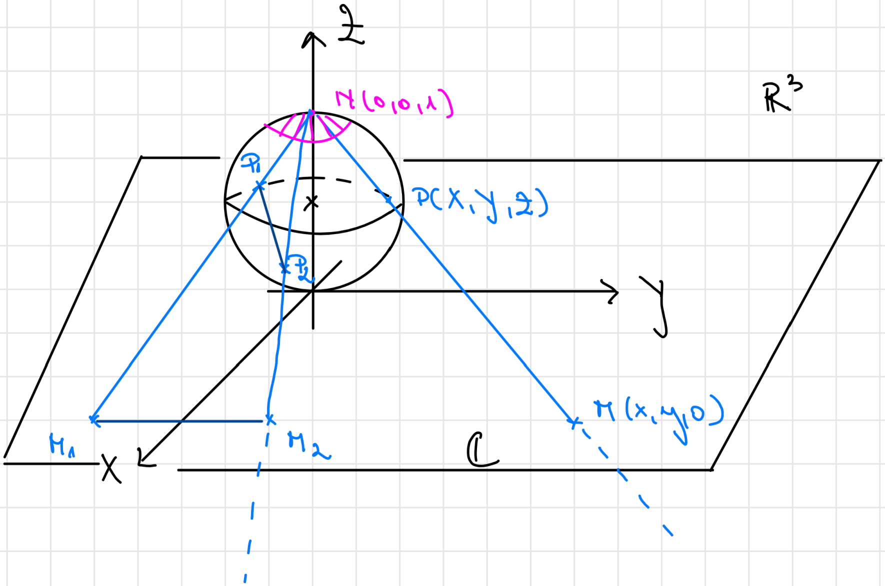
Def.

- $V\subseteq\mathbb{C}_\infty$ is a neighbourhood of ∞, if $\exists\ r>0$ s.t. $\{z\in\mathbb{C},\ |z|>r\}\sqcup\{\infty\}\subseteq V$.
- $V\subseteq\mathbb{C}_\infty$ is a neighbourhood of $z_0\in\mathbb{C}$, if $\exists\ r>0$ s.t. $\underbrace{\{z\in\mathbb{C},\ |z-z_0|<r\}}_{\mathcal{U}(z_0,r)}\subseteq V$



Notation:

For $z\in\mathbb{C}_\infty$, $\mathcal{V}(z)$ = the family of neighb. of z.



Def.

For $z_1, z_2 \in \mathbb{C}_\infty$, let $\underset{\hookrightarrow \text{chordal}}{d_c}(z_1, z_2) = \|P_1P_2\| =$

$$= \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}, \text{ where}$$

$P_j(X_j, Y_j, Z_j) = \tilde{\varphi}(z_j)$, $j = 1, 2$.

$d_c : \mathbb{C}_\infty \times \mathbb{C}_\infty \longrightarrow [0, \infty)$ is a metric.

Proposition:

$$d_c(z_1, z_2) = \begin{cases} \dfrac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \cdot \sqrt{1 + |z_2|^2}}, & z_1, z_2 \in \mathbb{C} \\[2ex] \dfrac{1}{\sqrt{1 + |z|^2}}, & (z_1 = \infty \ \& \ z_2 = z \in \mathbb{C}) \text{ or } (z_1 = z \in \mathbb{C} \ \& \ z_2 = \infty) \\[2ex] 0, & z_1 = z_2 = 0 \end{cases}$$

Remarks:

- $\forall z_0 \in \mathbb{C}_\infty$, $V \in \mathcal{V}(z) \iff \exists r>0$ s.t. $\{z \in \mathbb{C}_\infty,$
  $d_c(z, z_0) < r\} \subseteq V$.

- $V \in \mathcal{V}(\infty) \iff \tilde{\varphi}(V)$ is a neighbourhood of N on S.

- $\lim\limits_{n\to\infty} z_n = \infty \iff \lim\limits_{n\to\infty} |z_n| = \infty \iff d_c(z_n, \infty) \to 0$
  ↳ chordal distance

- $\lim\limits_{n\to\infty} z_n = 0 \iff \lim\limits_{n\to\infty} |z_n| = 0 \iff d_c(z_n, 0) \to 0$

- $(\mathbb{C}_\infty, d)$ is a complete metric space

Complex functions of a complex variable:

Let $A \subseteq \mathbb{C}$.

Def.

$f : A \to \mathbb{C}$ is called a complex function of a complex variable. We denote: $\operatorname{Re} f = u$ $\quad u, v : A \to \mathbb{R}$

$\operatorname{Im} f = v$ $\quad f = u + i\cdot v$,
$f(z) = u(z) + i\cdot v(z)$, $z \in A$

Def.

Let $z_0 \in A'$ (z_0 is an accumulation point for A) and $l \in \mathbb{C}$. f has the <u>limit l</u> at z_0, if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, s.t. $|f(z) - l| < \varepsilon$, $\forall\, 0 < |z - z_0| < \delta$, $z \in A$.

Remark:

- Notation: $\lim\limits_{z \to z_0} f(z) = l$
- $\lim\limits_{z \to z_0} f(z) = l \iff \lim\limits_{z \to z_0} \underbrace{u(z)}_{\operatorname{Re} f(z)} = \operatorname{Re} l$

$$\lim_{z \to z_0} \underbrace{v(z)}_{\operatorname{Im} f(z)} = \operatorname{Im} l$$

Def.

Let $z_0 \in A'$.

f has the <u>limit ∞</u> at z_0, if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, s.t. $|f(z)| > \varepsilon$, $\forall\, 0 < |z - z_0| < \delta$, $z \in A$.

Remark:

- Notation: $\lim_{z \to z_0} f(z) = \infty$

- $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$

Def.

Let $\infty \in A'$ $(\exists (z_n)_{n \in \mathbb{N}}$ in A s.t. $z_n \to \infty)$ and $l \in \mathbb{C}$.

f has the limit l at ∞, if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(z) - l| < \varepsilon, \forall |z| > \delta, z \in A$

Remark:

- Notation: $\lim_{z \to \infty} f(z) = l$

Def.

Let $\infty \in A'$. f has the limit ∞ at ∞, if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(z)| > \varepsilon, \forall |z| > \delta, z \in A$.

Notation:

$$\lim_{z \to \infty} f(z) = \infty$$

Remark:

$$\lim_{\substack{z \to z_0 \\ (z_0 \in \mathbb{C}_\infty)}} f(z) = \ell \in \mathbb{C}_\infty \iff \forall (z_n)_{n\in\mathbb{N}} \text{ in } A \text{ with}$$

$\lim_{n\to\infty} z_n = z_0$: $\lim_{n\to\infty} f(z_n) = \ell$.

Def.

Let $A \subseteq \mathbb{C}$, $z_0 \in A \cap A'$ and $f : A \to \mathbb{C}$.

f is continuous at z_0, if $\lim_{z\to z_0} f(z) = f(z_0)$.

f is continuous on A, if f is cont. at any point in A.

Remark:

- If $z_0 \in A$ is an isolated point, then f is cont. at z_0.
- The usual operations for cont. real functions hold for cont. complex functions.

Differentiability in $\mathbb{C}$

Def. 1

Let $J \subseteq \mathbb{R}$ be open, $f: J \longrightarrow \mathbb{C}$, $t_0 \in J$.

f is differentiable at t_0, iff $\exists \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} := f'(t_0) \in \mathbb{C}$

Remark 1:

$f = u + i \cdot v : J \longrightarrow \mathbb{C}$ is diff. at $t_0 \iff u, v : J \longrightarrow \mathbb{R}$ are diff. at t_0. Moreover, $f'(t_0) = u'(t_0) + v'(t_0)$

Let $G \subseteq \mathbb{C}$ be open.

Def. 2

$f: G \longrightarrow \mathbb{C}$, $z_0 \in G$. f is differentiable at z_0, iff $\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \in \mathbb{C}$.

$f'(z_0)$ is called the derivative of f at z_0.

Def. 3.

$f: G \to \mathbb{C}$, $z_0 \in G$. f is $\mathbb{C}$-differentiable at z_0,

if $\exists\, \alpha \in \mathbb{C}$, $\exists\, \omega : G \setminus \{z_0\} \to \mathbb{C}$ s.t. $\lim_{z \to z_0} \omega(z) = 0$

and $f(z) = f(z_0) + \alpha(z - z_0) + \omega(z) \cdot |z - z_0|$, $z \in G \setminus \{z_0\}$.

Proposition 1:

Let $f: G \to \mathbb{C}$, $z_0 \in G$. Then:

f is $\mathbb{C}$-diff. at $z_0 \iff f$ is diff. at z_0.

Proof.

f is $\mathbb{C}$-diff. at $z_0 \iff \exists\, \alpha \in \mathbb{C}$ s.t.

$$\frac{|z|}{|\zeta|} = \left|\frac{z}{|\zeta|}\right| = \left|\frac{z}{\zeta}\right|$$

$\hookrightarrow$ zeta

$$\lim_{z \to z_0} \underbrace{\left| \underbrace{\frac{f(z) - f(z_0) - \alpha(z - z_0)}{|z - z_0|}}_{\omega(z)} \right|} = 0.$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \alpha \right| \iff \exists\, \alpha \in \mathbb{C} \text{ s.t.}$$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \alpha$$

$\iff f$ is diff. at z_0

Remark 2:

$\alpha \in \mathbb{C}$ in Def. 3. is given by $\alpha = f'(z_0)$ from Def. 2.

LECTURE 4

24.X.2023

The Cauchy - Riemann theorem

Let $G \subseteq \mathbb{C}$ be open, $f = u + i \cdot v : G \longrightarrow \mathbb{C}$, $z_0 = x_0 + i \cdot y_0 \in G$
$(u = \operatorname{Re} f,\ v = \operatorname{Im} f)$

proposition

$\boxed{P1}$:

f is differentiable at z_0 $\Longleftrightarrow$ f is $\mathbb{C}$-differentiable at z_0

$$\left(\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \in \mathbb{C} \right)$$

$$\left(\exists\, \alpha \in \mathbb{C},\ \exists\, \omega : G \setminus \{z_0\} \longrightarrow \mathbb{C} \text{ with } \lim_{z \to z_0} \omega(z) = 0 \right.$$

$$\left. \text{s.t. } f(z) = f(z_0) + \alpha (z - z_0) + \omega(z) |z - z_0|,\ z \in G \setminus \{z_0\} \right)$$

Moreover, $\boxed{\alpha = f'(z_0)}$

Def. 1.

f is $\mathbb{R}$-differentiable at z_0, if u, v are Frechét differentiable at $(\underbrace{x_0, y_0}_{x_0 + i \cdot y_0})$

Obs.

Recall that $\mathbb{R}^2$ is identified with $\mathbb{C}$, $(x,y) = x+i\cdot y$, $\forall\, x,y\in\mathbb{R}$.

Remark 1:

i) $\|(x,y)\| = \sqrt{x^2+y^2}$, $\forall\,(x,y)\in\mathbb{R}^2=\mathbb{C}$ (the Euclidian norm on $\mathbb{R}^2$)

ii) f is $\mathbb{R}$-differentiable at $z_0 = x_0+i\cdot y_0$, if

$\exists\, a_1, b_1 \in \mathbb{R}$, $\exists\, \omega_1 : G\setminus\{(x_0,y_0)\} \longrightarrow \mathbb{R}$ with

$\lim\limits_{(x,y)\to(x_0,y_0)} \omega(x,y) = 0$ s.t.

$$u(x,y) = u(x_0,y_0) + a_1(x-x_0) + b_1(y-y_0) + \\ + \omega_1(x,y)\,\|(x-x_0,\, y-y_0)\|,\ (x,y)\in G\setminus\{(x_0,y_0)\}$$

and

$\exists\, a_2, b_2 \in \mathbb{R}$, $\exists\, \omega_2 : G\setminus\{(x_0,y_0)\} \to \mathbb{R}$ with

$\lim\limits_{(x,y)\to(x_0,y_0)} \omega_2(x,y) = 0$ s.t.

$$v(x,y) = v(x_0,y_0) + a_2(x-x_0) + b_2(y-y_0) + \\ + \omega_2(x,y)\,\|(x-x_0,\, y-y_0)\|,\ (x,y)\in G\setminus\{(x_0,y_0)\}$$

iii) $a_1 = \frac{\partial u}{\partial x}(x_0, y_0)$, $a_2 = \frac{\partial v}{\partial x}(x_0, y_0)$

$b_1 = \frac{\partial u}{\partial y}(x_0, y_0)$, $b_2 = \frac{\partial v}{\partial y}(x_0, y_0)$

iv) If $u, v \in \mathbb{C}^1(G)$ (continuous partial derivatives), then f is $\mathbb{R}$-differentiable on G

P2:

f is $\mathbb{R}$-differentiable at $z_0 = x_0 + i \cdot y_0 \iff \exists\, \alpha, \beta \in \mathbb{C}$,

$\exists\, \omega : G \setminus \{z_0\} \Rightarrow \mathbb{C}$ with $\lim\limits_{z \to z_0} \omega(z) = 0$ s.t.

$$f(z) = f(z_0) + \alpha(x - x_0) + \beta(y - y_0) + \omega(z)|z - z_0|,$$

$$\forall\, z = x + i \cdot y \in \mathbb{C} \setminus \{z_0\}$$

Moreover, $\alpha = \frac{\partial u}{\partial x}(x_0, y_0) + i \cdot \frac{\partial v}{\partial x}(x_0, y_0)$,

$\beta = \frac{\partial u}{\partial y}(x_0, y_0) + i \cdot \frac{\partial v}{\partial y}(x_0, y_0)$

Proof.

$\underset{u+i\cdot v}{\overset{\shortparallel}{f}}$ is $\mathbb{R}$-differentiable at $z_0 = x_0 + i\cdot y_0 \iff \exists\, a_1, b_1, a_2, b_2 \in \mathbb{R}$,

$\exists\, \omega_1, \omega_2 : G \setminus \{(x_0, y_0)\} \longrightarrow \mathbb{R}$ with

$$\lim_{(x,y)\to(x_0,y_0)} \omega_1(x,y) = \lim_{(x,y)\to(x_0,y_0)} \omega_2(x,y) = 0 \text{ s.t.}$$

$$\underbrace{u(x,y) + i\cdot v(x,y)}_{f(z)} = \underbrace{u(x_0, y_0) + i\cdot v(x_0, y_0)}_{f(z_0)} +$$

$$+ (a_1 + i\cdot a_2)(x - x_0) + (b_1 + i\cdot b_2)(y - y_0) +$$

$$+ \left(\omega_1(x,y) + \omega_2(x,y)\right) \underbrace{\|(x - x_0, y - y_0)\|}_{|z - z_0|}, \ \forall\, z = x + i\cdot y \in G \setminus \{z_0\}.$$

$$\xLeftrightarrow[\omega = \omega_1 + i\cdot\omega_2]{\alpha = a_1 + i\cdot a_2,\ \beta = b_1 + i\cdot b_2} \exists\, \alpha, \beta \in \mathbb{C},\ \exists\, \omega : G \setminus \{z_0\} \longrightarrow \mathbb{C}$$

with $\lim_{z\to z_0} \omega(z) = 0$ s.t.

$$f(z) = f(z_0) + \alpha(x - x_0) + \beta(y - y_0) + \omega(z)\cdot|z - z_0|,$$

$\forall\, \underset{x+i\cdot y}{\overset{\shortparallel}{z}} \in G \setminus \{z_0\}$

Using "Remark 1, iii)", we get the desired relations for α and β.

Theorem 1: (Cauchy – Riemann theorem)

f is differentiable at z_0 $\Longleftrightarrow$ f is $\mathbb{R}$-differentiable at z_0 and

$$\begin{cases} \dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0) \\[2ex] \dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0) \end{cases} \quad (*)$$

(these are Cauchy – Riemann equations of $f = u + i \cdot v$ at $z_0 = x_0 + i \cdot y_0$)

Proof.

$\boxed{\Longrightarrow}$ Assume that f is differentiable at z_0.

$\boxed{P_1}$ $\Rightarrow \exists\, \alpha = f'(z)$, $\exists\, \omega : G \setminus \{z_0\} \longrightarrow \mathbb{C}$ with

$\lim_{z \to z_0} \omega(z) = 0$ s.t.

$$f(z) = f(z_0) + \alpha (z - z_0) + \omega(z)\,|z - z_0|, \ \forall z \in G \setminus \{z_0\}$$

$$\xRightarrow[z_0 = x_0 + i \cdot y_0]{z = x + i \cdot y} f(z) = f(z_0) + \alpha (x - x_0) + i \cdot \alpha (y - y_0) + \omega(z)\,|z - z_0|, \ \forall z \in G \setminus \{z_0\}$$

$\boxed{P_2} \Longrightarrow f$ is $\mathbb{R}$-differentiable at z_0 and

$$\alpha = \frac{\partial u}{\partial x}(x_0, y_0) + i \cdot \frac{\partial v}{\partial x}(x_0, y_0),$$

$$\beta = i \cdot \alpha = \frac{\partial u}{\partial y}(x_0, y_0) + i \cdot \frac{\partial v}{\partial y}(x_0, y_0) \xRightarrow{\beta = i \cdot \alpha} (*)$$

$\boxed{\Longleftarrow}$ Assume that f is $\mathbb{R}$-differentiable at z_0 and $(*)$ holds.

$$\boxed{P_2} \Longrightarrow \exists\ \alpha = \frac{\partial u}{\partial x}(x_0, y_0) + i \cdot \frac{\partial v}{\partial x}(x_0, y_0),$$

$$\beta = \frac{\partial u}{\partial y}(x_0, y_0) + i \cdot \frac{\partial v}{\partial y}(x_0, y_0) \in \mathbb{C}$$

$\exists\ \omega : G \setminus \{z_0\} \longrightarrow \mathbb{C}$ with $\lim\limits_{z \to z_0} \omega(z) = 0$ s.t.

$$f(z) = f(z_0) + \alpha(x - x_0) + \beta(y - y_0) + \omega(z)|z - z_0|,$$
$$\forall\ z = x + i \cdot y \in G \setminus \{z_0\}$$

We observe that: $(*) \Longleftrightarrow \beta = i \cdot \alpha$

So, $f(z) = f(z_0) + \alpha(x - x_0) + i \cdot \alpha(y - y_0) + \omega(z)|z - z_0|$
$$\forall\ z = x + i \cdot y \in G \setminus \{z_0\}$$

$$\xRightarrow[z_0 = x_0 + i \cdot y_0]{z = x + i \cdot y} f(z) = f(z_0) + \alpha(z - z_0) + \omega(z)|z - z_0|, \forall z \in G \setminus \{z_0\}$$

$\boxed{P_1} \Longrightarrow f$ is differentiable at z_0.

ex.1. $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \bar{z}$, $z \in \mathbb{C}$ ($f = u + i \cdot v$, $u(x,y) = x$, $v(x,y) = -y$, $x + i \cdot y \in \mathbb{C}$)

f is $\mathbb{R}$-differentiable on $\mathbb{C}$ (by "Remark 1, iv"), but is NOT differentiable at any point in $\mathbb{C}$, because

(*) does NOT hold.

ex.2. $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} + i \dfrac{xy}{\sqrt{x^2+y^2}} , & z = x + i \cdot y \in \mathbb{C}^* \\ 0 , & z = 0 \end{cases}$

$f = u + i \cdot v$

u, v have partial derivatives on $\mathbb{C}$ which satisfy

(*) at $z_0 = 0$, but f is NOT differentiable at $z_0 = 0$.

Notations 1:

$$\frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \cdot \frac{\partial v}{\partial x}(x_0, y_0)$$

$$\frac{\partial f}{\partial y}(z_0) = \frac{\partial u}{\partial y}(z_0) = \frac{\partial u}{\partial y}(x_0, y_0) + i \cdot \frac{\partial v}{\partial y}(x_0, y_0)$$

Remark 2 :

From the proof of "Theorem 1", we have, if f is diff. at z_0, then :

$$\alpha = \boxed{f'(z_0) = \frac{\partial f}{\partial x}(z_0)} = -i\,\frac{\partial f}{\partial y}(z_0)$$

$(\beta = i\cdot\alpha \iff \alpha = -i\cdot\beta)$

Notations 2 :

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i\cdot\frac{\partial f}{\partial y}(z_0)\right)$$

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i\cdot\frac{\partial f}{\partial y}(z_0)\right)$$

$\boxed{P_3}$:

Let f be $\mathbb{R}$-differentiable at z_0. Then :

(*) $\iff$ $\boxed{\frac{\partial f}{\partial \bar{z}}(z_0) = 0}$

Moreover, if f is differentiable at z_0, then $\boxed{f'(z_0) = \frac{\partial f}{\partial z}(z_0)}$

Proof.

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i\cdot\frac{\partial f}{\partial y}(z_0)\right) \underset{\text{Notations 1}}{\overset{f=u+iv}{=\!=}}$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x}(x_0,y_0) - \frac{\partial v}{\partial y}(x_0,y_0) + i\cdot\left(\frac{\partial v}{\partial x}(x_0,y_0) + \right.\right.$$

$$\left.\left. + \frac{\partial u}{\partial y}(x_0,y_0)\right)\right] = 0$$

If f is diff. z_0, then $\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\Big(\underbrace{\frac{\partial f}{\partial x}(x_0,y_0)}_{\substack{\| \\ f'(z_0)}} -$ <u>Notations 2</u>

$$- \underbrace{i\cdot\frac{\partial f}{\partial y}(x_0,y_0)}_{\text{Remark 2} \quad f'(z_0)}\Big) = f'(z_0)$$

$\boxed{P_4}$:

If f is $\mathbb{R}$- differentiable at z_0, then

$\exists\ \omega : G\setminus\{z_0\} \longrightarrow \mathbb{C}$ with $\lim\limits_{z\to z_0} \omega(z) = 0$ s.t.

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)\cdot(z-z_0) + \frac{\partial f}{\partial \bar{z}}|z_0|(\overline{z-z_0}) +$$

$$+\,\omega(z)|z-z_0|,\ \forall\, z\in G\setminus\{z_0\}$$

LECTURE 5

31. X. 2023

$\boxed{P_1}$:

Let $f, g : G \longrightarrow \mathbb{C}$, $G \subseteq \mathbb{C}$ is open, be $\mathbb{R}$-differentiable at $z_0 \in G$. Then:

i) $\forall \alpha, \beta \in \mathbb{C}$, $\alpha f + \beta g$ is $\mathbb{R}$-differentiable and
$$\frac{\partial}{\partial z}(\alpha f + \beta g)(z_0) = \alpha \frac{\partial f}{\partial z}(z_0) + \beta \frac{\partial g}{\partial z}(z_0)$$

ii) $f \cdot g$ is $\mathbb{R}$-differentiable and
$$\frac{\partial}{\partial z}(f \cdot g)(z_0) = \frac{\partial f}{\partial z}(z_0) \cdot g(z_0) + f(z_0) \cdot \frac{\partial g}{\partial z}(z_0)$$
$$\frac{\partial}{\partial \bar{z}}(f \cdot g)(z_0) = \frac{\partial f}{\partial \bar{z}}(z_0) \cdot g(z_0) + f(z_0) \cdot \frac{\partial g}{\partial \bar{z}}(z_0)$$

iii) if f and g are differentiable at z_0, then:
$$(f \cdot g)'(z_0) = f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0)$$
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0) \cdot g(z_0) - f(z_0) \cdot g'(z_0)}{g^2(z_0)},$$
if $g(z_0) \neq 0$

$\boxed{P_2}$:

i) Let $G_1, G_2 \subseteq \mathbb{C}$ open, $f: G_1 \longrightarrow G_2$, $g: G_2 \longrightarrow \mathbb{C}$ be differentiable at z_0, respectively $f(z_0)$. Then:

$g \circ f$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0)$$

ii) If $f: G_1 \longrightarrow G_2$ is $\mathbb{R}$-differentiable and $g: G_2 \to \mathbb{C}$ differentiable at $f(z_0)$, then $g \circ f$ is $\mathbb{R}$-differentiable and

$$\frac{\partial}{\partial z}(g \circ f)(z_0) = g'(f(z_0)) \cdot \frac{\partial f}{\partial z}(z_0)$$

$$\frac{\partial}{\partial \bar{z}}(g \circ f)(z_0) = g'(f(z_0)) \cdot \frac{\partial f}{\partial \bar{z}}(z_0)$$

ex. 1.: $\frac{\partial z}{\partial z} = 1$, $\frac{\partial z}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}}{\partial z} = 0$, $\frac{\partial \bar{z}}{\partial \bar{z}} = -1$

$$\frac{\partial z^n}{\partial z} = n z^{n-1}, \quad n \in \mathbb{N}^*$$

$$\frac{\partial z^n}{\partial \bar{z}} = \frac{\partial \bar{z}^n}{\partial z} = 0$$

$$\frac{\partial \bar{z}^n}{\partial \bar{z}} = n \cdot \bar{z}^{n-1}$$

<u>ex. 2.</u> : $f : \mathbb{C} \longrightarrow \mathbb{C}$,

$$f(z) = \frac{1}{2} z^2 + \frac{1}{2} \bar{z}^2 + \underbrace{|z|^2}_{z \cdot \bar{z}} + z + \bar{z} + 1, \quad z \in \mathbb{C}$$

At which points is f differentiable?

f is differentiable at $z_0 \in \mathbb{C} \overset{\boxed{P_3}}{\underset{\text{Lecture 4}}{\Longleftrightarrow}}$ f is $\mathbb{R}$-differentiable at z_0 and $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$

Since $\operatorname{Re} f$, $\operatorname{Im} f$ are polynomial functions, $\underline{\operatorname{Re} f, \operatorname{Im} f \in C^1(\mathbb{C})}$, so f is $\mathbb{R}$-differentiable

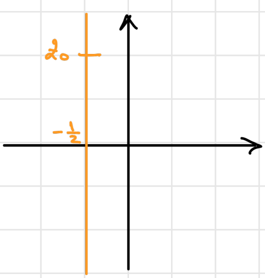
$$\frac{\partial f}{\partial z}(z_0) = \frac{\partial}{\partial \bar{z}}\left(\frac{1}{2} z^2 + \frac{1}{2} \bar{z}^2 + z \cdot \bar{z} + z + \bar{z} + 1\right)\Bigg|_{z = z_0} =$$

$$= \left(\frac{1}{2} \cdot 0 + \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot \bar{z} + z + 0 + 1 + 0\right)\Bigg|_{z = z_0} = \bar{z} + z_0 + 1$$

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \iff \underbrace{z_0 + \bar{z}_0}\, + 1 = 0$$

$$\iff \operatorname{Re} z_0 = -\frac{1}{2}$$

If $z_0 \in \mathbb{C}$ s.t. $\operatorname{Re} z_0 = -\frac{1}{2}$, then:



$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) =$$

(P_3, Lecture 4)

$$= \frac{\partial}{\partial z}\left(\frac{1}{2} z^2 + \frac{1}{2} \bar{z}^2 + z \cdot \bar{z} + z + \bar{z} + 1\right)\Bigg|_{z = z_0} =$$

$$= \left(\frac{1}{2}\cdot 2\cdot z + 0 + \bar{z} + 1 + 0 + 1\right)\Big|_{z=z_0} = z_0 + \overline{z_0} + 1 = 0$$

$$\operatorname{Re} z = -\frac{1}{2}$$

So, $f'(z_0) = 0$, if $\operatorname{Re} z_0 = -\frac{1}{2}$

Holomorphic function

Def. 1

Let $G \subseteq \mathbb{C}$ be open, $f: G \longrightarrow \mathbb{C}$.

We say that f is holomorphic at $z_0 \in \mathbb{C}$, if $\exists\, \mathcal{U}(z_0, r) \subseteq G$ s.t. f is differentiable at any $z \in G$.

$$\mathcal{H}(G) := \{ f: G \longrightarrow \mathbb{C},\ f \text{ is holomorphic} \}$$

ex. 2: let $f: \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = z\cdot \underbrace{|z|^2}_{z\cdot\bar{z}}$, $z \in \mathbb{C}$. Then f is differentiable at $z_0 = 0$, but f is NOT holomorphic at z_0.

f is $\mathbb{R}$-differentiable at z_0, because $\operatorname{Re} f, \operatorname{Im} f \in C^1(\mathbb{C})$ (polynomial function)

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(z^2\cdot\bar{z})\Big|_{z=z_0} = z_0^2 = 0 \Rightarrow f \text{ is diff. at } z_0$$

Since, $\dfrac{\partial f}{\partial \bar{z}}(z_0) = 0 \iff z_0 = 0$, f is NOT differentiable at any $z_0 \in \mathbb{C}^*$. So, f is NOT holomorphic at $z_0 = 0$.

Remark 1:

$$\mathcal{H}(G) \subsetneq \mathcal{C}(G)$$

Def. 2.

If $f \in \mathcal{H}(\mathbb{C})$, then f is called entire.

1. The complex polynomial function

$$p : \mathbb{C} \longrightarrow \mathbb{C}, \; p(z) = a_0 + a_1 z + \ldots + a_n z^n, \; z \in \mathbb{C},$$

where $a_0, \ldots, a_n \in \mathbb{C}$. Then $p \in \mathcal{H}(\mathbb{C})$ and

$$p'(z) = a_1 + 2a_2 z + \ldots + n \cdot a_n \cdot z^{n-1}, \; z \in \mathbb{C}.$$

2. The complex exponential function

Recall from the Seminar: $\lim \left(1 + \frac{z}{n}\right)^n = e^z$,

where $e^z = e^x(\cos y + i \cdot \sin y)$, $z = x + i \cdot y \in \mathbb{C}$

$\exp : \mathbb{C} \longrightarrow \mathbb{C}$, $\exp z = e^z$, $z \in \mathbb{C}$, is called <u>the complex exponential function</u>

<u>Remark 2:</u>

$$\exp \in \mathcal{H}(\mathbb{C}) \text{ and } \exp' = \exp \quad ((e^z)' = e^z,\ z \in \mathbb{C})$$

<u>Proof.</u>

$$\exp = u + i \cdot v \implies \boxed{\begin{aligned} u(x,y) &= e^x \cdot \cos y \\ v(x,y) &= e^x \cdot \sin y \end{aligned} \quad ,\ x + i \cdot y \in \mathbb{C}}$$

$u, v \in C^1(\mathbb{C}) \implies \exp$ is $\mathbb{R}$-diff. on $\mathbb{C}$

The Cauchy–Riemann system:

$$\begin{cases} \dfrac{\partial}{\partial x}(e^x \cdot \cos y) = \dfrac{\partial}{\partial y}(e^x \cdot \sin y) \\[2ex] \dfrac{\partial}{\partial y}(e^x \cdot \cos y) = -\dfrac{\partial}{\partial x}(e^x \cdot \sin y) \end{cases} \iff$$

$$\iff \begin{cases} e^x \cdot \cos y = e^x \cdot \cos y \\ e^x(-\sin y) = -e^x \cdot \sin y \end{cases}, \quad \forall x + iy \in \mathbb{C} \text{ „true"}$$

So, $\exp \in \mathcal{H}(\mathbb{C})$

$$\exp'(z) = (e^x)' = \frac{\partial \exp}{\partial x}(z) = \frac{\partial}{\partial x}\left(e^x(\cos y + i\cdot \sin y)\right) =$$
$$(z = x + i\cdot y)$$

$$= e^x(\cos y + i\cdot \sin y) = e^x = \exp z, \ \forall z = x + i\cdot y \in \mathbb{C}$$

Remark 3:

- $|e^z| = e^x, \quad \arg e^z = y \pmod{2\pi}, \ \forall z = x + i\cdot y$

- $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}, \ \forall z_1, z_2 \in \mathbb{C}$

- $e^{-z} = \dfrac{1}{e^z}, \ z \in \mathbb{C}$

$$\boxed{e^{x+i\cdot y} = e^x(\cos y + i \sin y)}$$

- $e^{z + (2k\pi) i} = e^z, \ z \in \mathbb{C}, \ \forall k \in \mathbb{Z}$

So, exp is a $(2\pi i)$ – periodic function on $\mathbb{C}$

- $z = x \in \mathbb{R} \Rightarrow e^z = e^x$ (real exponential function at $x \in \mathbb{R}$)

- (Euler's formulae) $\begin{cases} \cos y = \dfrac{e^{iy} + e^{-iy}}{2} \\ \sin y = \dfrac{e^{iy} - e^{-iy}}{2i} \end{cases}, y \in \mathbb{R}$

3. The complex trigonometric functions

$\cos, \sin : \mathbb{C} \longrightarrow \mathbb{C}$, $\boxed{\cos z = \dfrac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}}$

<u>Remark 4:</u>

- $\cos, \sin \in \mathcal{H}(\mathbb{C})$ and

  $\cos' = -\sin$, $\sin' = \cos$

- $\cos(z + 2k\pi) = \cos z$

  $\sin(z + 2k\pi) = \sin z$, $\forall z \in \mathbb{C}, k \in \mathbb{Z}$

4. The hyperbolic functions

$ch, sh : \mathbb{C} \to \mathbb{C}$, $ch\, z = \dfrac{e^{z} + e^{-z}}{2}$, $sh\, z = \dfrac{e^{z} - e^{-z}}{2}$, $z \in \mathbb{C}$

<u>Remark 5</u>

$ch, sh \in \mathcal{H}(\mathbb{C})$

$ch' = sh$, $sh' = ch$

A criterion for holomorphic functions to be constant

Recall: $\circ$ $\Delta \subseteq \mathbb{C}$ is a domain, if Δ is open and connected.

$\circ$ $f \in \mathcal{H}(G)$, $G \subseteq \mathbb{C}$ open $\Rightarrow$ f is continuous on G $(f \in \mathcal{C}(G))$

Theorem 1:

Let $\emptyset \neq \Delta \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(\Delta)$, $f = u + i \cdot v$.

The following are equivalent:

i) f is constant on Δ

ii) $f' = 0$ on Δ

iii) $\operatorname{Re} f$ is constant on Δ

iv) $\operatorname{Im} f$ is constant on Δ

v) $|f|$ is constant on Δ

ii) $\longrightarrow$ i)

$E = \{ z \in \Delta, f(z) = f(z_0)$

$z_0 \in \Delta$ - fixed

is open + closed in Δ $\}$

LECTURE 6 7. XI. 2023

A criterion for holomorphic functions to be constants

Theorem 1:

Let $\emptyset \neq \Delta \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(\Delta)$.

The following are equivalent:

i) f is constant on Δ

ii) $f' = 0$ on Δ

iii) $\operatorname{Re} f$ is constant on Δ

iv) $\operatorname{Im} f$ is constant on Δ

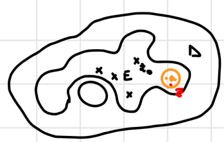
v) $|f|$ is constant on Δ

Proof.

i) $\Rightarrow$ ii), iii), iv), v)

$\boxed{\text{ii)} \Rightarrow \text{i)}}$ Assume that $f' = 0$ on Δ. Fix $z_0 \in \Delta$

and let $E = \{ z \in \Delta,\ f(z) = f(z_0) \}$



- E is closed in Δ: if $(z_n)_{n \in \mathbb{N}^*}$ is a sequence
(see lecture 2)
in E s.t. $z = \lim\limits_{n \to \infty} z_n \in \Delta$, then $z \in E$, because

since $f \in \mathcal{H}(\Delta) \Rightarrow f \in C(\Delta)$ and

$$f(z_n) = f(z_0), \ \forall n \in \mathbb{N}^*, \ f(z) = \lim_{n \to \infty} \underbrace{f(z_n)}_{f(z_0)} = f(z_0)$$

- $\underline{E \text{ is open in } \Delta:}$ Let $a \in E$. Since Δ is open (in $\mathbb{C}$),
(see Lecture 2)

$\exists\, \mathcal{U}(a, r) \subseteq \Delta$. Let $z \in \mathcal{U}(a, r)$. Note that

$$[a, z] = \{(1-t)a + t \cdot z, \ t \in [0,1]\} \subset \mathcal{U}(a, r)$$

We define: $\varphi : [0,1] \longrightarrow \Delta$, $\varphi(t) = (1-t)a + t \cdot z$,

$t \in [0,1]$

φ is differentiable at any $t \in (0,1)$ and

continuous on $[0,1]$, $\varphi([0,1]) = [a, z]$

Let $g = f \circ \varphi$. Since $f \in \mathcal{H}(\Delta)$, $\forall t_0 \in (0,1)$:

$$\lim_{t \to t_0} \frac{g(t) - g(t_0)}{t - t_0} =$$

$$= \lim_{t \to t_0} \underbrace{\frac{f(\varphi(t)) - f(\varphi(t_0))}{\varphi(t) - \varphi(t_0)}}_{\downarrow \ f'(\varphi(t_0))} \cdot \underbrace{\frac{\varphi(t) - \varphi(t_0)}{t - t_0}}_{= \underbrace{\frac{(1-t)a + tz - (1-t_0)a - t_0 z}{t - t_0}}_{z - a}} =$$

$$= \boxed{\underbrace{f'(\varphi(t_0))}_{=0}} \cdot (z - a) = 0$$

So, g is differentiable on $(0,1)$ with $g'=0$ on $(0,1)$ and g is continuous on $[0,1]$, hence g is constant on $[0,1]$ ($\operatorname{Re} g$, $\operatorname{Im} g$ are real functions on $[0,1]$ with vanishing derivatives on $(0,1)$)

Thus, $g(0) = g(1) \Rightarrow f(g(0)) = f(g(1)) \Rightarrow f(a) = f(z)$
$a \in E \Rightarrow f(a) = f(z_0)$ $\Rightarrow$

$\Rightarrow f(z) = f(z_0) \Rightarrow z \in E$

So, $U(a, r) \subseteq E$. So, E is open in Δ.

Since E is open + closed in Δ and Δ is connected, we have either $E = \emptyset$ or $E = \Delta$, but $E \neq \emptyset$, because $z_0 \in E$.

So, $E = \Delta \Rightarrow f$ is constant on Δ.

$\boxed{iii) \Rightarrow i)}$ Assume that $\operatorname{Re} f$ is constant on Δ.

Let $f = u + i \cdot v$, $u = \operatorname{Re} f$, $v = \operatorname{Im} f$

u is constant on $\Delta \Rightarrow \dfrac{\partial u}{\partial x} = 0$ on Δ and $\dfrac{\partial u}{\partial y} = 0$ on Δ

$f \in \mathcal{H}(\Delta) \xrightarrow[\text{theorem}]{\text{Cauchy-Riemann}} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 0$ on Δ

$f \in \mathcal{H}(\Delta) \Rightarrow f'(z) = \frac{\partial f}{\partial x}(x+i\cdot y) = \frac{\partial u}{\partial x}(x+i\cdot y) +$

$+ i \cdot \frac{\partial v}{\partial x}(x+i\cdot y) = 0, \forall z = x+i\cdot y$

So, using $\boxed{ii) \Rightarrow i)}$, f is constant on Δ.

$\boxed{iv) \Rightarrow i)}$ This case is similar the previous one.

$\boxed{v) \Rightarrow i)}$ Assume that $|f|$ is constant on Δ.

Then $u^2+v^2 = c$ on Δ, where $u = \operatorname{Re} f$, $v = \operatorname{Im} f$ and

$c \geq 0$ is a constant.

I. $c = 0 \Rightarrow u = v = 0 \Rightarrow f = 0$

II. $c \neq 0$

$\underline{u^2(x,y) + v^2(x,y)} = c, \forall x+i\cdot y \in \Delta \;\Big|\; \frac{\partial}{\partial x} \Big\} \Rightarrow 2u(x,y) \cdot \frac{\partial u}{\partial x}(x,y) +$

$+ 2v(x,y) \cdot \frac{\partial v}{\partial x}(x,y) = 0,$

$x+i\cdot y \in \Delta$

$\Big|\; \frac{\partial}{\partial y} \Big\} \Rightarrow 2u(x,y) \cdot \frac{\partial u}{\partial y}(x,y)$

$+ 2v(x,y) \cdot \frac{\partial v}{\partial y}(x,y) = 0,$

$x+i\cdot y \in \Delta$

$f \in \mathcal{H}(\Delta) \xRightarrow[\text{theorem}]{\text{Cauchy-Riemann}}$

$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ on Δ

So, $$\begin{cases} u(x,y) \underline{\dfrac{\partial u}{\partial x}(x,y)} + v(x,y) \cdot \underline{\dfrac{\partial v}{\partial x}(x,y)} = 0 \\ v(x,y) \cdot \underline{\dfrac{\partial u}{\partial x}(x,y)} - u(x,y) \cdot \underline{\dfrac{\partial v}{\partial x}(x,y)} = 0 \end{cases}, \ \forall\, x+i\cdot y \in \Delta$$

unknowns

$$\Longleftrightarrow \underbrace{\begin{bmatrix} u(x,y) & v(x,y) \\ v(x,y) & -u(x,y) \end{bmatrix}}_{A(x,y)} \begin{bmatrix} \dfrac{\partial u}{\partial x}(x,y) \\ \dfrac{\partial v}{\partial x}(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall\, x+i\cdot y \in \Delta$$

Since, $\det A(x,y) = -u^2(x,y) - v^2(x,y) = -c \neq 0$, $\forall\, x+i\cdot y \in \Delta$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ on } \Delta$$

$$f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 \text{ on } \Delta \xRightarrow{ii) \Rightarrow i)}$$

$\Rightarrow f$ is constant on Δ

<u>Remark</u>:

$\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dots + \dfrac{\partial^2}{\partial x_n^2}$ is called the Laplacian operator in $\mathbb{R}^n$.

Def. 1.:

Let $G \subseteq \mathbb{C}$ be open. $u : G \to \mathbb{R}$ is harmonic, if $u \in C^2(G)$ (u has cont. partial derivatives up to order 2) and
$$\Delta u(x,y) = \underline{\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0}, \quad \forall\, x+i\cdot y \in G$$
(the Laplace equations)

$\boxed{P_1}$:

Let $G \subseteq \mathbb{C}$ be open and $u \in C^2(G)$. If $\exists\, f \in \mathcal{H}(G)$ s.t. $u \in \operatorname{Re} f$, then u is harmonic.

Proof.:

Let $v = \operatorname{Im} f$. Then $f = u + i\cdot v \in \mathcal{H}(G)$

Cauchy-Riemann theorem. $\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$ on G $\Rightarrow$

$$\Rightarrow \Delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \Delta u = \underbrace{\frac{\partial^2 v}{\partial x \partial y}}_{\frac{\partial^2 u}{\partial x^2}} - \underbrace{\frac{\partial^2 v}{\partial y \partial x}}_{\frac{\partial^2 v}{\partial y^2}} = 0 \text{ on } G$$

(see Schwarz' criterion)

$$u \in C^2(G) \Rightarrow \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial y^2} \in C(G)$$

<u>ex. 1.</u>: $u : \mathbb{C} \to \mathbb{R}$, $u(x,y) = x^2 \cdot \cos y$, $x + iy \in \mathbb{C}$

Since $u = \operatorname{Re} \exp$ and $\exp \in \mathcal{H}(\Delta)$, $\boxed{P_1} \Rightarrow$
(exp = complex exponential function)
$\Rightarrow u$ is harmonic on $\mathbb{C}$

$\boxed{?}$ If $G \subseteq \mathbb{C}$ is open and $u : G \to \mathbb{R}$, $u \in C^2(G)$, does there exist $f \in \mathcal{H}(G)$ s.t. $\operatorname{Re} f = u$?

<u>Def. 2</u>

Let $G \subseteq \mathbb{C}$ be open and $P, Q : G \to \mathbb{R}$ be s.t. $P, Q \in C^1(G)$. Then, $\omega = P\,dx + Q\,dy$ is called a (linear) <u>differential form of class C^1 on G</u>:

$\forall z \in G$, $\omega(z) = P(z) \cdot dx + Q(z) \cdot dy$ is a linear function from $\mathbb{R}^2$ to $\mathbb{R}$, where

$dx(x,y) = x$, $dy(x,y) = y$, $\forall (x,y) \in \mathbb{R}^2$

ω is : • <u>closed</u>, if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ on G

$\Uparrow$

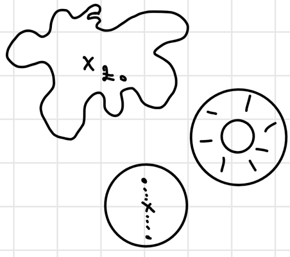
• <u>exact</u>, if $\exists f : G \to \mathbb{R}$, $f \in C^1(G)$ s.t.

$$df = \underbrace{\frac{\partial f}{\partial x}}_{P} dx + \underbrace{\frac{\partial f}{\partial y}}_{Q} dy = \omega$$

$\boxed{P_2}$: (Poincaré)

Let $\Delta \subseteq \mathbb{C}$ be a <u>starlike</u> domain w.r.t. $z_0 \in \Delta$ and ω be a differential form of class C^1 on Δ.

Then, ω is closed $\Longleftrightarrow$ ω is exact.



<u>Theorem 1</u>:

Let $\Delta \subseteq \mathbb{C}$ be a <u>starlike</u> domain w.r.t. $z_0 \in \Delta$ and <u>$u : \Delta \to \mathbb{R}$ be harmonic</u>.

Then $\exists f \in \mathcal{H}(\Delta)$ s.t. $\operatorname{Re} f = u$

Proof.

We look for a function on $v : \Delta \to \mathbb{R}$, $v \in C^2(\Delta)$, s.t.

u, v satisfy the Cauchy – Riemann system:

$$(*) \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

on Δ. If this holds, we can take

$f = u + i \cdot v$ and then $f \in \mathcal{H}(\Delta)$

Let $\boxed{P = -\dfrac{\partial u}{\partial y}}$, $\boxed{Q = \dfrac{\partial u}{\partial x}}$ on Δ.

u is harmonic $\Rightarrow$ $$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial Q}{\partial x} \text{ on } \Delta$$

$\Rightarrow \omega = P\,dx + Q\,dy$ is closed

$\xRightarrow[\Delta \text{ starlike}]{\boxed{P_2}}$ ω is exact $\Rightarrow \exists\, v : \Delta \to \mathbb{R}$,

$v \in C^2(\Delta)$ s.t. $P = \dfrac{\partial v}{\partial x}$, $Q = \dfrac{\partial v}{\partial y}$

$\Rightarrow -\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$,

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow (*)$ holds

LECTURE 7

14. XI. 2023

The Cauchy-Riemann system in polar coordinates

$\boxed{P_1}$:

Let $G \subseteq \mathbb{C}^*$ be open and $\tilde{G} \subseteq (0,\infty) \times \mathbb{R}$ be

s.t. $= \{ (r \cdot \cos\theta, r \cdot \sin\theta), (r, \theta) \in \tilde{G} \}$

Let $f : G \to \mathbb{C}$, $f = u + i \cdot v$ where $u = \operatorname{Re} f$, $v = \operatorname{Im} f$

Let $\tilde{u}(r,\theta) = u(r\cos\theta, r \cdot \sin\theta)$

$\tilde{v}(r,\theta) \quad v(r\cos\theta, r\sin\theta)$, $(r,\theta) \in \tilde{G}$

If $\tilde{u}, \tilde{v} \in C^1(G)$ then:

$$f \in \quad (G) \iff \begin{cases} \dfrac{\partial \tilde{u}}{\partial r}(r,\theta) = \dfrac{1}{r} \dfrac{\partial \tilde{v}}{\partial \theta}(r,\theta) \\ \dfrac{\partial \tilde{u}}{\partial}(r,\theta) = -r \dfrac{\partial \tilde{v}}{\partial r}(r,\theta) \end{cases}, \forall (r,\theta) \in \tilde{G}$$

(*)

Proof:

$\tilde{u}, \tilde{v} \in C^1(\tilde{G}) \Rightarrow u, v \in C^1(G)$ (The Inverse Function theorem implies)

$u, v \in C^1(G) \Rightarrow f$ is $\mathbb{R}$-diff on G

By the Cauchy-Riemann Theorem,

$$f \in \mathcal{H}(G) \iff \boxed{\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) &= -\frac{\partial v}{\partial x}(x,y) \end{aligned} \quad , \forall\, x+iy \in G}$$

(**)

$$\iff \begin{cases} \dfrac{\partial u}{\partial x}(r\cos\theta, r\sin\theta) = \dfrac{\partial v}{\partial y}(r\cos\theta, r\sin\theta) \\ \dfrac{\partial v}{\partial y}(r\cos\theta, r\sin\theta) = -\dfrac{\partial v}{\partial x}(r\cos\theta, r\sin\theta) \end{cases} , \forall\, (r,\theta) \in \tilde{G}$$

Let $(r,\theta) \in \tilde{G}$

$$\underbrace{\frac{\partial \tilde{u}}{\partial r}(r,\theta)}_{\frac{\partial \tilde{u}}{\partial r}} = \frac{\partial}{\partial r}\left(u(r\cos\theta, r\sin\theta)\right) \overset{\text{chain}}{\underset{\text{rule}}{=\!=}} {}^{*Ai}$$

Notation $\rightarrow$

$$= \underbrace{\frac{\partial u}{\partial x}(r\cos\theta, r\sin\theta)}_{=\frac{\partial u}{\partial x}} \underbrace{\frac{\partial}{\partial r}(r\cos\theta)}_{\cos\theta} + \underbrace{\frac{\partial u}{\partial y}(r\cos\theta, r\sin\theta)}_{=\frac{\partial u}{\partial y}} \underbrace{\frac{\partial}{\partial r}(r\sin\theta)}_{\sin\theta}$$

Notation :

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \sin\theta$$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x} \cdot (-r\sin\theta) + \frac{\partial u}{\partial y} (r\cos\theta)$$

$$\frac{\partial \tilde{v}}{\partial r} = \frac{\partial v}{\partial x} \cdot \cos\theta + \frac{\partial v}{\partial y} \cdot \sin\theta$$

$$\frac{\partial \tilde{v}}{\partial \theta} = \frac{\partial v}{\partial x} (-r\sin\theta) + \frac{\partial v}{\partial y} (r\cos\theta)$$

$$(*) \iff \begin{cases} \dfrac{\partial u}{\partial x} \cdot \cos\theta + \dfrac{\partial u}{\partial y} \cdot \sin\theta = \dfrac{1}{r}\left(\dfrac{\partial v}{} \cdot (-r \cdot \sin\theta) + \dfrac{\partial v}{\partial y}(r \cdot \cos\theta)\right), \\[2ex] \dfrac{\partial u}{\partial x} \cdot (-r \cdot \sin\theta) + \dfrac{\partial u}{\partial y} \cdot (r \cdot \cos\theta) = -r\left(\dfrac{\partial v}{\partial x} \cdot \cos\theta + \dfrac{\partial v}{\partial y} \cdot \sin\theta\right) \end{cases}$$

$$\iff \begin{cases} \left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right) \cdot \cos\theta + \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \cdot \underline{\qquad} = 0 \\[2ex] \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) \cdot (-\sin\theta) + \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \cdot \cos\theta = 0 \end{cases} \iff$$

$$\iff \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $\det \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$

$(*) \iff (**)$

The complex logarithm

Def. 1.

$$\mathrm{Log} : \mathbb{C}^* \longrightarrow \mathcal{P}(\mathbb{C})$$

$$\log z = \ln|z| + i \cdot \mathrm{Arg}\, z$$

$$= \{ \ln|z| + i\cdot(\arg z + 2k\pi),\ k \in \mathbb{Z} \ ,\ z \in \mathbb{C}^*$$

Log is c   the multi valued complex logarithm.

/ ark 1:

1. $[w \in \log z \iff e^{w} = z]$ $(\forall z \in \mathbb{C}^*\ \ w \in \mathbb{C})$
2. $\mathrm{Log}(z_1, z_2) = \log z_1 + \log z_2$, $\forall z_1, z_2 \in \mathbb{C}^*$
3. $\log\left(\frac{1}{z}\right) = -\log z$, $\forall z \in \mathbb{C}^*$

Def. 2.

Let $D \subseteq \mathbb{C}^*$ be a dom   , $f : D \longrightarrow \mathbb{C}$ is a <u>branch</u> of log on D, if $f \in \mathcal{H}(D)$ and $f(z) \in \log z$, $\forall z \in D$

$\boxed{P_2}$:

Let $\Delta \subseteq \mathbb{C}^*$ be a domain

i) If φ is a branch of Log on Δ, then $\varphi'(z) = \frac{1}{z}$, $z \in \Delta$

ii) If φ_1, φ_2 are bran of log on Δ, then

$$\exists\, k_0 \in \mathbb{Z} \text{ s.t. } \varphi_2(z) = \varphi_1(z) + (2k\pi)\cdot i,\ \forall z \in \Delta$$

Moreover, if, in addition, $\exists\, z_0 \in \Delta$ s.t. $\varphi_1(z_0) = \varphi_2(z_0)$,

then $\varphi_1 = \varphi_2$.

proof.:

i) $\varphi \in \mathcal{H}(\Delta)$ and $\varphi(z) \in \log z$, $\forall z \in \Delta$

$$\Big\Updownarrow \text{ / ork 1,}$$

$$e^{\varphi(z)} = z$$

$$\Rightarrow (e^{\varphi(z)})' = z',\ \forall z \in \Delta \Rightarrow \underbrace{e^{\varphi(z)}}_{z} \cdot \varphi'(z) = 1,\ \forall z \in \Delta \Rightarrow$$

$$\Rightarrow \varphi'(z) = \frac{1}{z},\ \forall z \in \Delta$$

ii) $\varphi_1(z) \in \underbrace{\log z}_{\ln|z| + i\cdot \mathrm{Arg}\, z}$, $\forall z \in \Delta \Leftrightarrow \forall z \in \Delta$, $\exists\, k_1(z) \in \mathbb{Z}$ s.t.

$$\varphi_1(z) = \ln|z| + i\cdot(\arg z + 2k_1(z)\pi)$$

$\varphi_2 \in \log z$, $\forall z \in \Delta \Leftrightarrow \forall z \in \Delta$, $\exists\, k_2(z) \in \mathbb{Z}$ s.t.

$f_2(z) = \ln|z| + i\cdot(\arg z + 2k_2(z)\pi)$

Let $f = f_1 - f_2 \in \mathcal{H}(D)$, $f'(z) = f_1'(z) - f_2'(z) \overset{i)}{=} \frac{1}{z} - \frac{1}{z} = 0$,

$\forall z \in D$

Since D is a dom , f is constant (by Theorem 1, Lecture 5)

We h , $f(z) = (\ln|z| + i\cdot(\arg z + 2k_1(z)\pi)) -$

$-(\ln|z| + i\cdot(\arg z + 2k_2(z)\pi) = 2(\overbrace{k_1(z) - k_2(z)})\pi\cdot i$,

$\forall z \in D$

$k_1 - k_2$ is constant $\Rightarrow \exists k_0 \in \mathbb{Z}$ s.t. $k_2(z) = k_1(z) + k_0$

$\forall z \in D$

$\boxed{P_3}$:

Let $B = \{z \in \mathbb{C}, -\pi < \operatorname{Im} z < \pi\}$

$D = \{z \in \mathbb{C}, z \notin (-\infty, 0]\} = \mathbb{C} \setminus (-\infty, 0]$

and $f: B \longrightarrow D$, $f(z) = e^z$, $z \in$

T f is bijective, $f^{-1}(w) = \ln|w| + i\cdot\arg w$, $w \in D$

and $f^{-1} \in \mathcal{H}(D)$

<u>Proof</u>.

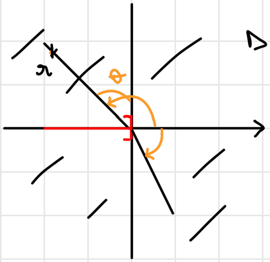
f is bijective and $f^{-1}(w) = \ln|w| + i\cdot\arg w$, $w \in D$

see Seminar 6.

We use $\boxed{P_1}$: $\varphi^{-1} = u + i \cdot v : D \longrightarrow B$, let

$$\tilde{D} = \{(r,\theta),\ r>0,\ \theta \in (-\pi,\pi)\}$$

$$D = \{(r\cos\theta,\ r\sin\theta),\ (r,\theta)\in\tilde{D}\}$$



$$\tilde{u}(r,\theta) = u(r\cos\theta,\ r\sin\theta)$$

$$= \ln|r\cos\theta + i\sin\theta| = \ln r$$

$$\tilde{v}(r,\theta) = v(r\cos\theta,\ r\sin\theta) \ , \ \forall (r,\theta)\in\tilde{D}$$

$$= \arg(r(\cos\theta + i\cdot \qquad)) = \theta$$

(see $\boxed{P_1}$ ✱) $\iff$ $\begin{cases} \dfrac{\partial\tilde{u}}{\partial r}(r,\theta) = \dfrac{1}{r} = \dfrac{1}{r}\dfrac{\partial\tilde{v}}{\partial\theta}(r,\theta) = \dfrac{1}{r} \ (\\[2ex] \dfrac{\partial\tilde{u}}{\partial\theta}(r,\theta) = 0 = -r\cdot\dfrac{\partial\tilde{v}}{\partial r}(r,\theta) = 0 \ \text{(true)} \end{cases}$

So, $\varphi^{-1} \in \mathcal{H}(D)$

<u>Def. 3.</u>

$$B = \{z\in D,\ -\pi < \operatorname{Im} z < \pi\}$$

$$D = \mathbb{C}\setminus(-\infty, 0]$$

<u>log</u> : $D \longrightarrow B$, $\log z := \ln|z| + i\cdot\arg z$, $z\in D$

is c the principal branch of log on D.

Remark 2:

1. $\log$ is the unique branch of Log on Δ with $\log 1 = 0$

2. $(\log z)' = \frac{1}{z}$, $z \in \Delta$

3. Let $k \in \mathbb{Z}$, $\log_k : \Delta \longrightarrow \mathbb{C}$, $\log_k = \log + 2\pi k i$ (principal) is the unique branch of Log on Δ with $\log_k 1 = 2k\pi i$

$\log_k(\Delta) = 2k\pi i + B$

The ometric interpretation of the complex derivative

Let $G \subseteq \mathbb{C}$ be open, $a, b \in \mathbb{R}$, $a < b$.

Def. 4.

Let $\gamma : [a,b] \longrightarrow G$ be a continuous function

γ is called a path (in G)

$\{\gamma\} = \gamma([a,b]) =$ support of γ

γ is smooth, if $\gamma \in C^1([a,b])$

Let $f \in \mathcal{H}(G)$ be s.t. $f'(z) \neq 0$, $z \in G$

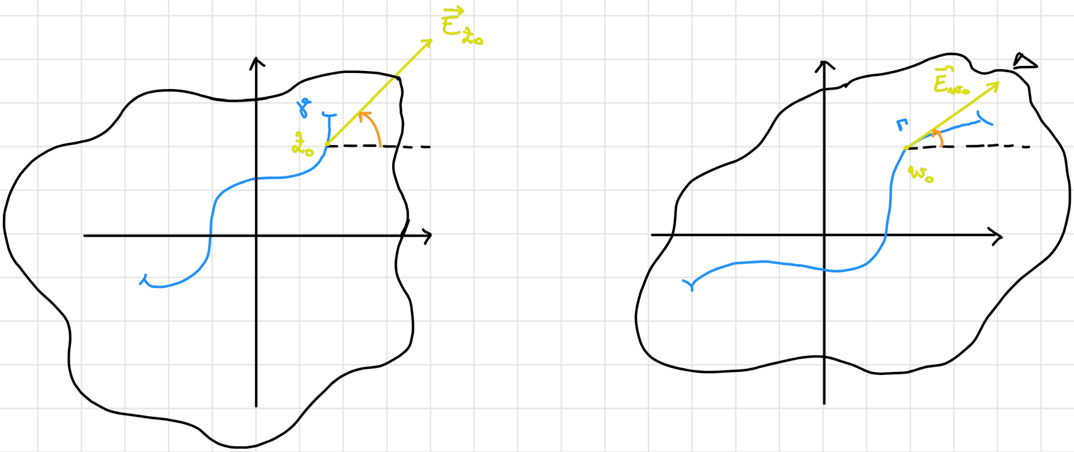
$\underline{\Delta = f(G)}$ fix $z_0 \in G$, $w_0 = f(z_0) \in \Delta$

Let be a smooth path in G s.t. $z_0 \in \{\gamma\}$

and $\gamma'(t_0) \neq 0$, where $t_0 \in [a,b]$ s.t. $\gamma(t_0) = z_0$

$w_0 \in \{\Gamma\}$ and $\Gamma'(t_0) \overset{(1)}{=} f'(z_0) \cdot \gamma'(t_0)$

Let $\vec{E}_{z_0} = \gamma'(t_0)$, $\vec{E}_{w_0} = \Gamma'(t_0)$ be tangent vectors of γ and Γ at z_0 and w_0.



(1) $\Rightarrow |\vec{E}_{w_0}| = \boxed{k_{z_0}} \cdot |\vec{E}_{z_0}|$,

where k_{z_0} = <u>coefficient of linear deformation at z_0</u>

k_0 is independent of γ : $\boxed{k_{z_0} = |f'(z_0)|}$

$\{z \in \mathbb{C} : |f'(z)| < 1\}$ is <u>the set contracted by f</u>

$\{z \in \mathbb{C} : |f'(z)| > 1\}$ is <u>the set expanded by f</u>

$(1) \Rightarrow \operatorname{Arg} \Gamma'(t_0) = \arg f'(z_0) + \operatorname{Arg} \gamma'(t_0)$

Let $\theta_{z_0} = \arg f'(z_0) =$ <u>the angle of rotation of $\vec{E}_{z_0}$ through</u>

$\arg \vec{E}_{w_0} = \theta_{z_0} + \arg \vec{E}_{z_0} \pmod{2\pi}$

LECTURE 8 28.XI.2025

Paths in the complex plane

Def. 1.

Let $[a,b] \subseteq \mathbb{R}$, $n \in \mathbb{N}^*$.

A division of $[a,b]$ is a sequence $\Delta = (t_0, t_1, \dots, t_n)$ s.t.

$a = t_0 < t_1 < \dots < t_n$.

Let $\gamma : [a,b] \longrightarrow \mathbb{C}$

$V(\gamma, \Delta) = \sum_{i=1}^{n} |\gamma(t_n) - \gamma(t_{n-1})|$ is called the variation of γ w.r.t. Δ

$Div[a,b] = \{\Delta,\ \Delta$ is a division of $[a,b]\}$

$V(\gamma) = \sup_{\Delta \in Div[a,b]} V(\gamma, \Delta)$ is called the total variation of γ

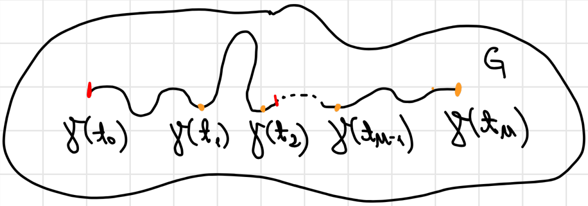
γ is said to be with bounded variation if $V(\gamma) < \infty$.

Def. 2.

Let $G \subseteq \mathbb{C}$. $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a path in G, if

$\{\gamma\} = \gamma([a,b]) \subseteq G$, γ is continuous on $[a,b]$ and

$\exists\ \Delta = (t_0, t_1, \dots, t_n) \in \mathcal{D}iv\,[a,b]$ s.t. γ has continuous derivative on $[t_{k-1}, t_k]$, $\forall\, k \in \{1, \dots, n\}$ (however, the lateral derivati at $t_1, \dots, t_{n-1}$ may differ)



<u>Remark 1:</u>

Any path γ in $\mathbb{C}$ is with bounded variation.

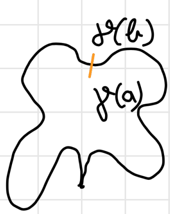
Moreover, if Δ is as in Def. 2. for γ, then

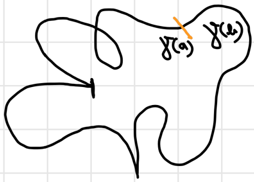
$$\ell(\gamma) = \sum_{k=1}^{n} \int_{t_{n-1}}^{t_k} |\gamma'(t)|\, dt$$ is called the <u>length</u> of γ.

<u>Def. 3.</u>

A path $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a <u>contour</u>, if $\gamma(a) = \gamma(b)$.

If, in addition, $\gamma|_{[a,b)}$ is bijective, then γ is <u>Jordan contour</u>.



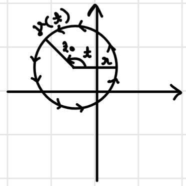


Example 1:

Let $x_0 \in \mathbb{C}$, $r > 0$, $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$, $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then γ is a Jordan contour with $\{\gamma\} = \partial U(z_0, r)$

Remark 2:

If $G \subseteq \mathbb{C}$ is open, $f \in \mathcal{H}(G)$ and γ is a path in G, then $f \circ \gamma$ is a path in $f(G)$ and $(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$ for every t s.t. γ has a derivative at t.

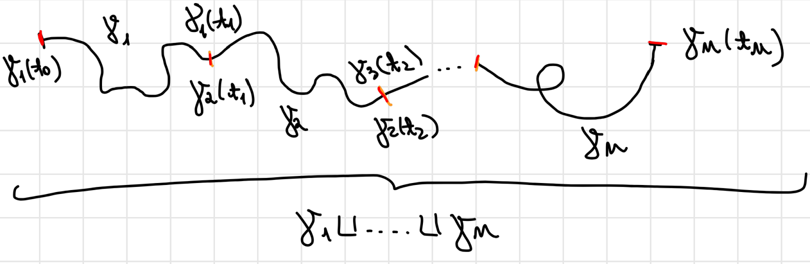


Def. 4.

Let $\Delta = (t_0, t_1, \dots, t_n) \in \mathrm{Div}[a, b]$ and

$\gamma_k : [t_{k-1}, t_k] \longrightarrow \mathbb{C}$ be a path, $\forall\, k \in \overline{1,n}$ s.t.

$\gamma_k(t_k) = \gamma_{k+1}(t_k)$, $k = \overline{1, n-1}$



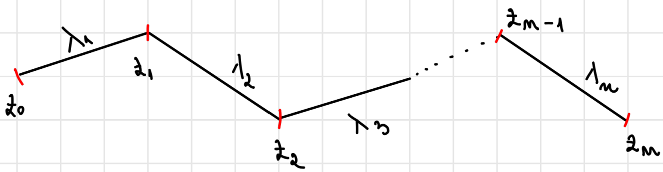
$\gamma_1 \sqcup \ldots \sqcup \gamma_m$ given by $(\gamma_1 \sqcup \ldots \sqcup \gamma_m)(t) = \gamma_k(t)$, $t \in [t_{k-1}, t_k]$, $k = \overline{1,m}$ is called the union path of $\gamma_1, \ldots, \gamma_m$.

Example 2:

Let $z_0, z_1, \ldots, z_m \in \mathbb{C}$, $\Delta = (t_0, t_1, \ldots, t_m) \in \mathrm{Div}[a,b]$,

$$\lambda_k : [t_{k-1}, t_k] \longrightarrow \mathbb{C}, \quad \lambda_k(t) = \frac{t_k - t}{t_k - t_{k-1}} \cdot z_{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} \cdot z_k,$$

$$t \in [t_{k-1}, t_k]$$



$\lambda_1 \sqcup \ldots \sqcup \lambda_m$ is polygonal path,

$$\{\lambda_1 \sqcup \ldots \sqcup \lambda_m\} = [z_0, z_1] \sqcup [z_1, z_2] \sqcup \ldots \sqcup [z_{m-1}, z_m]$$

Def. 5.

If $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a path, then $\gamma^- : [a,b] \longrightarrow \mathbb{C}$ given by $\gamma^-(t) = \gamma(a+b-t)$, $t \in [a,b]$ is called the opposite path of γ.

Def. 6

Let $\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$, $\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ be paths.

γ_1 and γ_2 are equivalent, if $\exists\, h : [a_1, b_1] \longrightarrow [a_2, b_2]$ s.t.

h is continuous, strictly increasing, $h([a_1, b_1]) = [a_2, b_2]$

and $\gamma_1 = \gamma_2 \circ h$.

Note that, if γ_1 and γ_2 are equivalent, then $\{\gamma_1\} = \{\gamma_2\}$

Complex integral

Def. 7

Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a path and $f : \{\gamma\} \longrightarrow \mathbb{C}$ be continuous.

The (complex) <u>integral</u> of f along γ is the Riemann-Stötjes integral: $\displaystyle\int_a^b f(\gamma(t))\, d\gamma(t)$.

We denote it by $\displaystyle\int_\gamma f$ or $\displaystyle\int_\gamma f(z)\, dz$.

Remark 3:

If $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a path that has the division Δ as in Def. 2., then: $\int_\gamma f = \int_\gamma f(z)\, dz =$

$$= \int_a^b f(\gamma(t))\, d\gamma(t) = \sum_{k=1}^{n} f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

Example 3:

Let $z_0 \in \mathbb{C}$, $r > 0$, $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$, $\gamma(t) = z_0 + re^{it}$,
$t \in [0, 2\pi]$

$$f : \mathbb{C} \setminus \{z_0\},\ f(z) = \frac{1}{z - z_0},\quad z \in \mathbb{C} \setminus \{z_0\}$$

$$\int_\gamma f = \int_\gamma f(z)\, dz = \int_\gamma \frac{1}{z - z_0}\, dz$$

$$\gamma'(t) = (z_0 + re^{it})'\Big|_t = r(e^{it})'\Big|_t \overset{\text{Remark 2}}{=\!=} r \cdot \underbrace{\exp'}_{\exp}(it) \cdot (it)'\Big|_t$$

$$= r \cdot e^{it} \cdot i,\ t \in [0, 2\pi]$$

$$\int_\gamma \overbrace{\frac{1}{z - z_0}}^{f(z)}\, dz \overset{\text{Remark 3}}{=\!=} \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} \frac{1}{\gamma(t) - z_0} \cdot \gamma'(t)\, dt$$

$$= \int_0^{2\pi} \frac{1}{\cancel{z_0} + \cancel{re^{it}} - \cancel{z_0}} \cdot (\cancel{re^{it}} \cdot i)\, dt = \int_0^{2\pi} i\, dt = \underline{\underline{2\pi i}}$$

Properties :

- $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$, $\forall \alpha, \beta \in \mathbb{C}$,
  $f, g : \{\gamma\} \to \mathbb{C}$ continuous

- $\int_{\gamma^-} f = - \int_\gamma f$

- γ_1 and γ_2 are equivalent paths $\Rightarrow \int_{\gamma_1} f = \int_{\gamma_2} f$,
  $\forall f : \underset{\{\gamma_2\}}{\underset{\shortparallel}{\{\gamma_1\}}} \longrightarrow \mathbb{C}$

- $\int_{\gamma_1 \sqcup \dots \sqcup \gamma_m} f = \int_{\gamma_1} f + \dots + \int_{\gamma_m} f$

- $\left| \int_\gamma f \right| \le \max_{z \in \{\gamma\}} |f(z)| \cdot \ell(\gamma)$

- If $(f_n)_{n \in \mathbb{N}}$ is a sequence of continuous functions from $\{\gamma\}$ to $\mathbb{C}$, that <u>converges uniformly</u> to a continuous function f from $\{\gamma\}$ to $\mathbb{C}$, then $\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f$

Def. 8.

Let $G \subseteq \mathbb{C}$ be open and $f : G \to \mathbb{C}$

$F : G \to \mathbb{C}$ is a primitive of f, if $F \in \mathcal{H}(G)$ and $F' = f$.

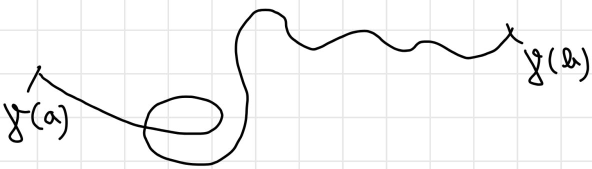
Theorem 1: (the connection between the primitive and the integral)

Let $\Delta \subseteq \mathbb{C}$ be a domain and $f : \Delta \to \mathbb{C}$ be continuous.

Then: f has a primitive on $\Delta \iff \int_\gamma f = 0, \ \forall \gamma$ a contour in Δ

Moreover, if F is a primitive of f, then

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)), \ \forall \gamma : [a,b] \to \Delta \text{ a path.}$$



Example 4:

$f : \mathbb{C}^* \longrightarrow \mathbb{C}, \ f(z) = \frac{1}{z}, \ z \in \mathbb{C}^*$

$f \in \mathcal{H}(\mathbb{C}^*)$, but f has no primitive on $\mathbb{C}^*$.

Proof.

Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}, \ \gamma(t) = e^{it}, \ t \in [0, 2\pi]$

Example 3 $\Rightarrow \int_\gamma f = \int_\gamma \frac{1}{z-0}\, dz = 2\pi i \neq 0 \xRightarrow{\text{Theorem 1}}$

$\Rightarrow f$ has <u>NO</u> primitive

<u>Example 5:</u>

$f: \mathbb{C} \setminus (-\infty, 0]$

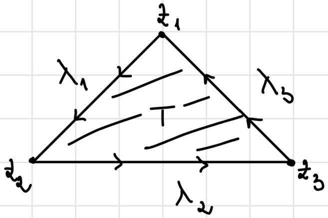
LECTURE 9 : 5. XI . 2023

Complex integrals and primitives

Theorem 1: (Cauchy's theorem for triangles)

Let $z_1, z_2, z_3 \in \mathbb{C}$, $\lambda_1 : [0,1] \longrightarrow \mathbb{C}$

$$\lambda_1(t) = (1-t)z_1 + (t-0)z_2, \; t \in [0,1]$$



$$\lambda_2 : [1,2] \longrightarrow \mathbb{C}$$

$$\lambda_2(t) = (2-t)z_2 + (t-1)z_3, \; t \in [1,2]$$

$$\lambda_3 : [2,3] \longrightarrow \mathbb{C}$$

$$\lambda_3(t) = (3-t)z_3 + (t-2)z_2, \; t \in [2,3]$$

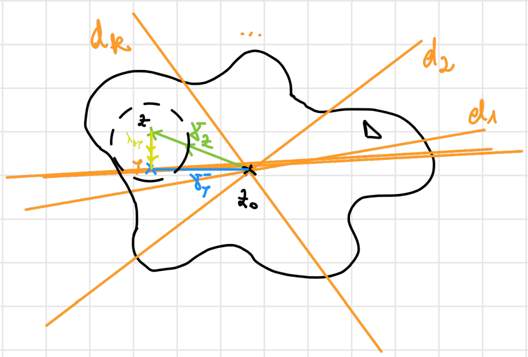
$\Delta = \lambda_1 \sqcup \lambda_2 \sqcup \lambda_3 : [0,3] \longrightarrow \mathbb{C}$ is the union path of $\lambda_1, \lambda_2, \lambda_3$

If T is the convex hull of z_1, z_2, z_3 (T is the triangle with vertices z_1, z_2, z_3) and $f : T \longrightarrow \mathbb{C}$ is continuous on T and holomorphic on $\operatorname{int} T$, then $\int_\Delta f = 0$.

Theorem 2:

Let $\Delta \subseteq \mathbb{C}$ be a starlike domain w.r.t. $z_0 \in \Delta$ and $d_1, d_2, \dots, d_k$ be lines in $\mathbb{C}$ passing through z_0.

If $d = \bigsqcup_{j=1}^{k} d_j$ and $f : \Delta \longrightarrow \mathbb{C}$ is continuous on Δ and holomorphic on $\Delta \setminus d$, then f has a primitive on Δ.



Proof:

For $z \in \Delta$, let $\gamma_z : [0,1] \longrightarrow \mathbb{C}$, $\gamma_z(t) = (1-t)z_0 + t \cdot z$, $t \in [0,1]$

γ_z is a path in Δ, because Δ is starlike w.r.t. z_0.

Let $g : \Delta \longrightarrow \mathbb{C}$, $g(z) = \int_{\gamma_z} f$, $z \in \Delta$.

We want to prove that g is a primitive for f:

$$\forall z \in \Delta, \quad g'(z) = \lim_{\gamma \to z} \frac{g(z) - g(\gamma)}{z - \gamma} = f(z)$$

Fix arbitrary $z \in \Delta$. For $\gamma \in \Delta$, let $\lambda_{z\gamma} : [1,2] \longrightarrow \mathbb{C}$

$$\lambda_{z\gamma}(t) = (2-t)z + (t-1)\gamma,$$
$$t \in [1,2]$$

There exists $r > 0$ s.t. $\mathcal{U}(z,r) \subseteq \Delta$ (Δ is open)

Then $\Delta_\varphi = \cancel{\gamma_z} \sqcup \lambda_{z\gamma} \sqcup \gamma_\varphi$ is a path in Δ, $\forall \varphi \in \mathcal{U}(z,r)$

By $\lfloor \overline{T_1} \rfloor$, we have $\int_{\Delta_\varphi} f = 0$, $\forall \varphi \in \mathcal{U}(z,r)$ (homework for details)

Let $\gamma \in \mathcal{U}(z,r)$,

$$0 = \int_{\Delta_\varphi} f = \int_{\gamma_z} f + \int_{z\varphi} f + \int_{\gamma_\varphi^-} f =$$

$$= g(z) + \int_1^2 f(\lambda_{z\varphi}(t)) \cdot \lambda'_{z\varphi}(t)\, dt - g(\gamma) =$$

$$= g(z) - g(\gamma) + \int_1^2 f((2-t)z + (t-1)\gamma) \cdot (\gamma - z)\, dt$$

$$\overset{\gamma \neq z}{\Longrightarrow} \quad \frac{g(z) - g(\gamma)}{z - \gamma} = \int_1^2 f((2-t)z + (t-1)\gamma)\, dt$$

$$f((2-t)z + (t-1)\gamma) \xrightarrow[\substack{\text{uniformly} \\ \text{w.r.t. } t \in [0,1] \\ \text{when } \gamma \to z}]{} f(z)$$

Using the properties of the complex integral, we deduce that $\lim\limits_{\gamma \to z} \dfrac{g(z) - g(\gamma)}{z - \gamma} = f(x)$

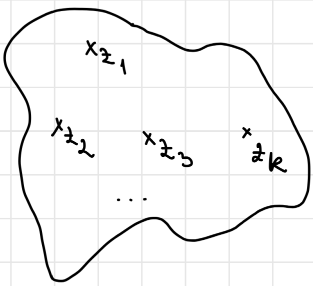
Theorem 3:

Let Δ be a starlike domain w.r.t. $z_0 \in \Delta$ and $\{z_1, z_2, \dots, z_k\} \subseteq \Delta$. If $f : \Delta \longrightarrow \mathbb{C}$ is continuous on Δ and holomorphic on $\Delta \setminus \{z_1, \dots, z_k\}$, then f has primitives on Δ.

In particular, $\int_\gamma f = 0$, $\forall\, \gamma$ a contour in Δ.

Proof.

Use T_2 and [T_1 from Lecture 8]



Theorem 4: (fundamental theorem of Cauchy)

Let Δ be a starlike domain w.r.t. $z_0 \in \Delta$.

If $f \in \mathcal{H}(\Delta)$, then f has primitives on Δ.

In particular, $\int_\gamma f = 0$, $\forall\, \gamma$ contour in Δ, $\forall f \in \mathcal{H}(\Delta)$.

Remark 1:

A domain $\Delta \subseteq \mathbb{C}$ is called simply connected, if $\mathbb{C}_\infty \setminus \Delta$ is connected in $\mathbb{C}_\infty$ („Δ has no holes").

$\boxed{T_4}$ holds for Δ being simply connected.

Cauchy's integral formula

$\boxed{P_1}$:

Let γ be a contour in $\mathbb{C}$, $\varphi : \{\gamma\} \longrightarrow \mathbb{C}$ be continuous.

and $\underbrace{\phi}_{\text{capital } \varphi} : \mathbb{C} \setminus \{\gamma\} \longrightarrow \mathbb{C}$, $\phi(z) = \int_\gamma \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta$, $z \in \mathbb{C} \setminus \{\gamma\}$

Then ϕ is infinitely (complex) differentiable on $\mathbb{C} \setminus \{\gamma\}$

and $\phi^{(n)} = n! \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$, $z \in \mathbb{C} \setminus \{\gamma\}$, $n \in \mathbb{N}$.

(derivative of order n)

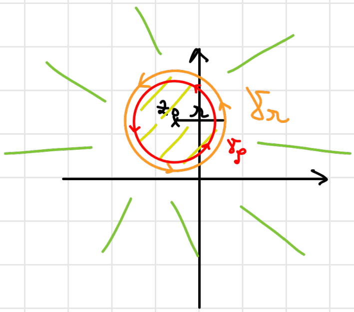
Remark 2:

$$\overbrace{\frac{\partial^n}{\partial z^n} \left(\frac{1}{\zeta - z} \right)}^{\left(\frac{1}{\zeta - z} \right)^{(n)} \Big|_z} = n! \frac{1}{(\zeta - z)^{n+1}}, \quad n \in \mathbb{N}, \ \zeta \neq z$$

Corollary 1:

Let $z_0 \in \mathbb{C}$, $r > 0$, $\gamma_r : [0, 2\pi] \longrightarrow \mathbb{C}$, $\gamma_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$

Then
$$\int_{\gamma_r} \frac{1}{\zeta - z} \, d\zeta = \begin{cases} 2\pi i, & \text{if } z \in U(z_0, r) \\ 0, & \text{if } z \in \mathbb{C} \setminus \overline{U}(z_0, r) \end{cases}$$



Theorem 5: (Cauchy's integral formula)

Let $z_0 \in \mathbb{C}$, $r > 0$, $f : \overline{U}(z_0, r) \longrightarrow \mathbb{C}$ be continuous on $\overline{U}(z_0, r)$ and holomorphic on $U(z_0, r)$.

Then f is infinitely differentiable on $U(z_0, r)$ and

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in U(z_0, r)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta, \quad z \in U(z_0, r),$$

where $\gamma_r : [0, 2\pi] \longrightarrow \mathbb{C}$, $\gamma_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$

Proof:

Let $z \in U(z_0, r)$ and $g_z : \overline{U}(z_0, r) \longrightarrow \mathbb{C}$

$$g_z(\zeta) = \begin{cases} \dfrac{f(z) - f(\zeta)}{z - \zeta}, & \zeta \in \overline{U}(z_0, r) \setminus \{z\} \\ f'(z), & \zeta = z \end{cases}$$

We want to prove that:

$$\int_{\gamma_r} g_z = 0.$$

For $\rho \in (0, r)$, let $\gamma_\rho : [0, 2\pi] \longrightarrow \mathbb{C}$

$$\gamma_\rho(t) = z_0 + \rho e^{it}, \quad t \in [0, 2\pi]$$

$\boxed{T_3} \Rightarrow \int_{\gamma_\rho} g_z = 0$, $\forall \rho \in (0, r)$, because g_z is <u>continuous</u> on <u>$U(z_0, r)$</u> and <u>holomorphic</u> on <u>$U(z_0, r) \setminus \{z\}$</u>

$$0 = \int_{\gamma_\rho} g_z = \underbrace{\int_0^{2\pi} g_z(z_0 + \rho e^{it}) \underbrace{(\rho \cdot i \cdot e^{it})}_{\gamma_\rho'(t)} \, dt}$$

$\downarrow$ uniformly w.r.t. $t \in [0, 2\pi]$ when $\rho \nearrow r$ (g_z is cont. on $\overline{U}(z_0, r)$)

$$\underbrace{\int_0^{2\pi} g_z(z_0 + r e^{it}) (r \cdot i e^{it}) \, dt}_{\int_{\gamma_r} g_z}$$

So, $\int_{\gamma_r} g_z = 0$.

$$0 = \int_{\gamma_r} g_z = \int_{\gamma_r} \frac{f(z)-f(\zeta)}{z-\zeta}\, d\zeta = f(z) \underbrace{\int_{\gamma_r} \frac{1}{z-\zeta}\, d\zeta}_{= 2\pi i \ \boxed{C_1}} - \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$\Longrightarrow$ (1)

Let $n \in \mathbb{N}$, $\varphi : \{\gamma_r\} \longrightarrow \mathbb{C}$

$$\varphi(\zeta) = f(\zeta), \quad \zeta \in \{\gamma_r\} = \partial U(z_0, r)$$

$$\phi : \mathbb{C} \setminus \{\gamma_r\} \longrightarrow \mathbb{C}$$

$$\phi(z) = \int_{\gamma_r} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta = \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \mathbb{C}\setminus\{\gamma_r\}$$

$$\boxed{P_1} \Longrightarrow \phi^{(n)}(z) = n! \int_{\gamma_r} \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta =$$

$$= \int_{\gamma_r} \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad z \in \mathbb{C}\setminus\{\gamma_r\}$$

But,

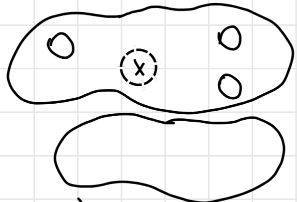
LECTURE 10:

12. XII. 2023

Applications of the Cauchy formula

Theorem 1:

Let $G \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(G)$. Then f is infinitely differentiable on G.



Proof:

Use locally (i.e., in every disk in G) the Cauchy formula theorem. (T_5, Lecture 9)

Theorem 2: (Morera)

Let $G \subseteq \mathbb{C}$ be open. If $f : G \longrightarrow \mathbb{C}$ has primitives on G, then $f \in \mathcal{H}(G)$.

Proof.

$$\exists\, F \in \mathcal{H}(G) \text{ s.t. } F' = f \xRightarrow{T_1} f \in \mathcal{H}(G)$$

Theorem 3:

Let $\Delta \subseteq \mathbb{C}$ be a <u>starlike</u> domain and $f : \Delta \longrightarrow \mathbb{C}$.

Then, f has primitives on $\Delta \iff f \in \mathcal{H}(\Delta)$.

Proof.

"$\Rightarrow$" true by T_2

"$\Leftarrow$" true by T_4, Lecture 9

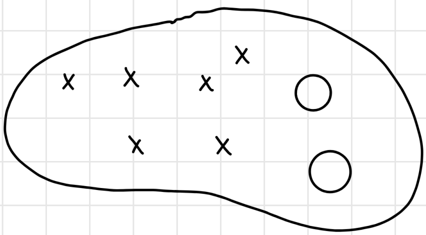
Theorem 4:

Let $G \subseteq \mathbb{C}$ be open, $\{z_1, \dots, z_n\} \subset G$ and $f : G \longrightarrow \mathbb{C}$ be continuous on G and holomorphic on $G \setminus \{z_1, \dots, z_n\}$.

Then, $f \in \mathcal{H}(G)$.

Proof.

Use T_3, Lecture 9.



Theorem 5: (Cauchy's inequalities)

Let $z_0 \in \mathbb{C}$, $r > 0$, $f : \overline{U}(z_0, r) \longrightarrow \mathbb{C}$ be continuous on $\overline{U}(z_0, r)$ and holomorphic $U(z_0, r)$. Then:

$$|f^{(n)}(z_n)| \leq \frac{n!}{r^n} \cdot M, \ \forall n \in \mathbb{N}, \text{ where } M = \max_{\partial U(z_0, r)} |f|$$

Proof.

Use Cauchy's formulae theorem, T_5, lecture 9, and the properties of the complex integral (use lecture 8).

Theorem 6: (Liouville)

If f is entire ($f \in \mathcal{H}(\mathbb{C})$) and bounded, then f is constant.

Proof.

$T_5 \Rightarrow \forall z \in \mathbb{C}, \forall r > 0, |f'(z)| \leq \frac{1}{r} \cdot M$, where

$M = \sup_{\mathbb{C}} |f| < \infty$.

$r \to \infty \Rightarrow \forall z \in \mathbb{C}, f'(z) = 0 \xRightarrow[\text{Lecture 6}]{T_1} f$ is constant.

Theorem 7: (the fundamental theorem of Algebra)

Let $p : \mathbb{C} \longrightarrow \mathbb{C}$, $p(z) = a_n \cdot z^n + \ldots + a_1 z + a_0$, $z \in \mathbb{C}$,

where $a_0, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$, $n \in \mathbb{N}^*$.

Then $\exists z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$.

Proof.

Assume that $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Then $f = \frac{1}{p} \in \mathcal{H}(\mathbb{C})$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{1}{a_n z^n + \ldots + a_1 z + a_0} =$$

$$= \lim_{\rho \to \infty} \underbrace{\frac{1}{z^n}}_{\to 0} \frac{1}{a_n + a_{n-1}\underbrace{\left(\frac{1}{z}\right)}_{\to 0} + \ldots + a_1\underbrace{\left(\frac{1}{z^{n-1}}\right)}_{\to 0} + a_0\underbrace{\left(\frac{1}{z^n}\right)}_{\to 0}} =$$

$$= 0 \cdot \frac{1}{a_n} = 0$$

So, $\forall \varepsilon > 0$, $\exists r > 0$ s.t. $|f(z)| < \varepsilon$, $\forall |z| > r$.

$\Rightarrow \exists r > 0$ s.t. $|f(z)| < 1$, $\forall z \in \mathbb{C} \setminus \overline{U}(0,r)$

$f \in \mathcal{H}(\mathbb{C}) \Rightarrow f$ is cont. on $\overline{U}(0,r) \Rightarrow f$ is bounded on $\overline{U}(0,r)$ $\Rightarrow$

$\Rightarrow f$ is hol. and bounded $\xrightarrow{T6}$ f is constant, contradictio

Sequences of holomorphic functions and power series

Def. 1.

Let $G \subseteq \mathbb{C}$ be open and $f_n : \mathbb{C} \longrightarrow \mathbb{C}$, $n \in \mathbb{N}$

$(f_n)_{n \in \mathbb{N}}$ converges uniformly on compacts of G to $f : G \longrightarrow G$ (notation: $f_n \overset{u.c.}{\underset{G}{\Longrightarrow}} f$), if

$\forall K \subset G$, K compact: $(f_n)_{n \in \mathbb{N}}$ converges uniformly on K to f: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon$, $\forall z \in K$, $\forall n \in \mathbb{N}$.

Theorem 8: (Cauchy – Haddamard theorem)

Consider the power series around $z_n \in \mathbb{C}$:

$$(*) \quad \sum_{n=0}^{\infty} a_n (z-z_0)^n, \text{ where } a_n \in \mathbb{C},\ n \in \mathbb{N} \text{ w.r.t.}$$

the variable $z \in \mathbb{C}$.

Let $R \in [0, \infty]$ given by

$$(**) \quad \frac{1}{R} = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}, \text{ called}$$

the radius of convergence of $(*)$

If $R \in (0, \infty]$, then $(*)$ converges uniformly on compacts of $\mathcal{U}(z_0, R)$ to a holomorphic function

$$S : \mathcal{U}(z_0, R) \longrightarrow \mathbb{C},\ S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,\ z \in \mathcal{U}(z_0, R)$$

Moreover, $S'(z) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (z-z_0)^{n-1},\ z \in \mathcal{U}(z_0, R)$.

Remark 1:

i) If $R = \infty$, then $\mathcal{U}(z_0, R)$ becomes $\mathbb{C}$ in T_8.

ii) If $R \in (0, \infty]$, then $a_n = \frac{S^{(n)}(z_0)}{n!}$, $n \in \mathbb{N}$, so

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = S(z) = \sum_{n=0}^{\infty} \frac{S^{(n)}(z_0)}{n!} (z-z_0)^n,$$

$z \in \mathcal{U}(z_0, R)$.

Theorem 9: (Taylor power series expansion)

Let $z_0 \in \mathbb{C}$, $r>0$, $f \in \mathcal{H}(\mathcal{U}(z_0, r))$. Then:

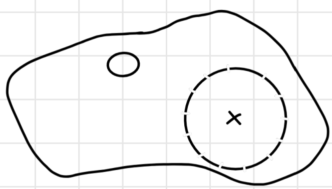
$\exists!$ power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with radius of convergence $R \geq r$ s.t. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $z \in \mathcal{U}(z_0, r)$

Moreover, $a_n = \dfrac{f^{(n)}(z_0)}{n!}$, $n \in \mathbb{N}$

Remark 2:

Let $G \subseteq \mathbb{C}$ be open, $f : G \longrightarrow \mathbb{C}$.

$f \in \mathcal{H}(G) \iff f$ has a power series expansion around every point in G (i.e., f is analytic)



ex. 1.

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Then $f \in C^{\infty}(\mathbb{R})$ (infinitely differentiable on $\mathbb{R}$), but it is NOT analytic, in real case.

Classical power series expansions (around 0)

1. $$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^n + \dots, \quad z \in U(0,1)$$

2. $$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots, \quad z \in \mathbb{C}$$

3. $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + \frac{(-1)^n}{(2n+1)!} \cdot z^{2n+1} + \dots, \quad z \in \mathbb{C}$$

4. $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + \frac{(-1)^n}{(2n)!} \cdot z^{2n} + \dots, \quad z \in \mathbb{C}$$

5. $$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots + (-1)^{n+1} \frac{z^n}{n} + \dots, \quad z \in U(0,1)$$

Zeros of holomorphic functions.

Def. 2.

Let $G \subseteq \mathbb{C}$ open, $f \in \mathcal{H}(\mathbb{C})$, $z_0 \in G$.

z_0 is a zero of f, if $f(z_0) = 0$.

z_0 is a zero of order $n \in \mathbb{N}^*$ of f,

if $f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0$, $f^{(n)}(z_0) \neq 0$.

Theorem 10: (Factorization of holomorphic functions)

Let $G \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(G)$, $z_0 \in G$, $n \in \mathbb{N}^*$.

Then, z_0 is a zero of order $n \iff \exists\ \varphi \in \mathcal{H}(\mathbb{C})$ s.t.

$\varphi(z_0) \neq 0$ and $\boxed{f(z) = (z-z_0)^n \cdot \varphi(z)}$, $z \in G$.

Proof.

"$\Leftarrow$" If $\exists\ \varphi \in \mathcal{H}(\mathbb{C})$ s.t. $\varphi(z) \neq 0$, $f(z) = (z-z_0)^n \cdot \varphi(z)$, $z \in G$,

then $f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0$ and

$f^{(n)}(z_0) = n!\, \varphi(z_0) \neq 0$.

So, z_0 is zero of order n of f.

"$\Rightarrow$" If $f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0$, $f^{(n)}(z_0) \neq 0$,

then, by Tg (Taylor theorem),

$$f(z) = \sum_{k=n}^{\infty} a_k (z-z_0)^n, \quad z \in U(z_0, r),$$

where $r > 0$ s.t. $U(z_0, r) \subseteq G$

$$\text{So, } f(z) = (z-z_0)^n \sum_{k=n}^{\infty} a_k (z-z_0)^{k-n}, \quad z \in U(z_0, r)$$

By Tg, $\sum\limits_{k=n}^{\infty}$

LECTURE 12: 19. XII. 2023

The identity theorem

Theorem 1:

Let $G \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(G)$, $z_0 \in G$ and $m \in \mathbb{N}^*$.

Then: z_0 is a zero of order $m \iff \exists\, \varphi \in \mathcal{H}(G)$ s.t.

$\varphi(z_0) \neq 0$ and $f(z) = (z - z_0)^m \cdot \varphi(z)$, $\forall\, z \in G$

Proof: (continuation)

$$\text{Let } \varphi(z) = \begin{cases} \tilde{\varphi}(z), & z \in U(z_0, r) \\ \cancel{\dfrac{f(z)}{(z-z_0)}}, & z \in G \setminus U(z_0, r) \end{cases}$$

φ satisfies the conditions that we need

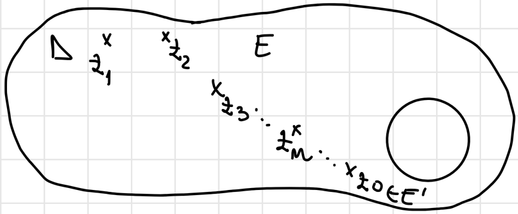
Theorem 2: (Identity theorem)

Let $\Delta \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(\Delta)$. Then, the following are equivalent:

i) $f \equiv 0$

ii) $\exists\, a \in \Delta$ s.t. $f^{(n)}(a) = 0$, $\forall\, n \in \mathbb{N}$

iii) $\exists\, E \subseteq \Delta$ s.t. $E' \cap \Delta \neq \emptyset$ and $f|_E \equiv 0$



(E' = set of <u>accumulation</u> points,

$z_0 \in E' \iff \exists\, (z_n)_{n \in \mathbb{N}^*}$ in $E \setminus \{z_0\}$ s.t.

$$\lim_{n \to \infty} z_n = z_0 \quad)$$

<u>Proof</u>.

Clearly, i) $\Rightarrow$ ii) and i) $\Rightarrow$ iii)

$\boxed{ii \Rightarrow i}$ Let $A = \{ z \in D ,\ f^{(n)}(z) = 0 ,\ \forall n \in \mathbb{N} \}$

ii) $\Rightarrow A \neq \emptyset$

- <u>A is open in D</u>: Let $a \in A \xrightarrow{D \text{ open}} \exists\, r > 0$ s.t. $U(a,r) \subseteq D$

$$f \in \mathcal{H}(D) \xrightarrow[\text{theorem, Lecture 10}]{\text{Taylor}} f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (z-a)^n =$$

$$\underset{\substack{f^{(n)}(a) = 0, \\ \forall n \in \mathbb{N}}}{\overset{a \in A}{=}} 0 \ ,\ \forall z \in U(a,r)$$

$\Rightarrow U(a,r) \subset A$. Since a is arbitrary in A, A is open in D.

- <u>A is closed in Δ</u>: Let $(z_n)_{n\in\mathbb{N}^*}$ in A be such that $\lim\limits_{n\to\infty} z_n = z_0 \in \Delta$

Fix $n\in\mathbb{N}$, $f\in\mathcal{H}(\Delta) \Rightarrow f^{(n)} \in \mathcal{H}(\Delta) \Rightarrow$

$f^{(n_0)} \in C(\Delta) \Rightarrow f^{(n_0)}(z_0) = \lim\limits_{n\to\infty} f^{(n_0)}(z_n) \xRightarrow{z_n\in A,\, n\in\mathbb{N}}$

So, $z_0\in A \Rightarrow A$ is closed in Δ.

$$\left.\begin{array}{l} A \text{ is open \& closed in } \Delta \\ \Delta \text{ is connected} \end{array}\right\} \Rightarrow A=\emptyset \text{ or } A=\Delta \xRightarrow{A\neq\emptyset} A=\Delta \Rightarrow f\equiv 0$$

$\boxed{iii \Rightarrow ii}$ $\exists E\subseteq \Delta$ s.t. E in $\Delta \neq \emptyset$ and $f|_E \equiv 0$

So, $\exists$ $(z_n)_{n\in\mathbb{N}^*}$ in $E\setminus\{z_0\}$ s.t. $\lim\limits_{n\to\infty} z_n = z_0 \in E'\cap\Delta$

Assume that, $\exists$ $n_0\in\mathbb{N}^*$ s.t. $f(z_0)=\dots=f^{(n_0-1)}(z_0)$, $f^{(n_0)}(z_0)\neq 0$

$T_1 \Rightarrow \exists\, \varphi\in\mathcal{H}(\Delta)$ s.t. $\varphi(z_0)\neq 0$ and

$$f(z) = (z-z_0)^{n_0}\cdot \varphi(z),\ \forall z\in\Delta$$

$\forall n\in\mathbb{N}$: $\underbrace{f(z_n)}_{=0} = \underbrace{(z_n-z_0)}_{\neq 0,\ z_n\in E\setminus\{z_0\}}^{n_0} \varphi(z_n) \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow \forall n\in\mathbb{N},\ \varphi(z_n)=0 \\ \varphi\in\mathcal{H}(\Delta)\Rightarrow \varphi\in C(\Delta) \end{array}\right\} \Rightarrow$$

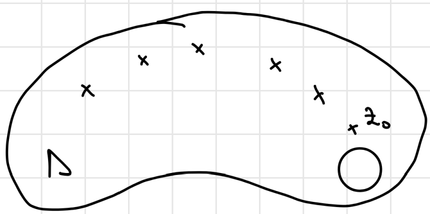
$$\Rightarrow \lim_{n\to\infty} \underbrace{\varphi(z_n)}_{=0} = \varphi(z_0) \Rightarrow \varphi(z_0) \equiv 0$$

contradiction!

Corollary 1:

If $\Delta \subseteq \mathbb{C}$ is a domain and $f \in \mathcal{H}(\Delta)$, then:

$f \equiv 0$ on $\Delta \iff \exists\ z_0 \in \Delta$ and $(z_n)_{n\in\mathbb{N}^*}$ in $\Delta \setminus \{z_0\}$ s.t. $\lim_{n\to\infty} z_n = z_0$ and $f(z_n) = 0,\ \forall n \in \mathbb{N}^*$



Corollary 2:

If $\Delta \subseteq \mathbb{C}$ is a domain and $f \in \mathcal{H}(\Delta)$ s.t. $f \not\equiv 0$, then every zero of f is isolated.

Corollary 3:

If $\Delta \subseteq \mathbb{C}$ is a domain and $f, g \in \mathcal{H}(\Delta)$ s.t. $f|_E = g|_E$, where $E \subseteq \Delta$, $E' \cap \Delta \neq \emptyset$, then $f = g$ on Δ.

Theorem 3: (Maximum modulus theorem)

i) If $\Delta \subseteq \mathbb{C}$ is a domain, $f \in \mathcal{H}(\Delta)$ and

$\exists z_0 \in \Delta$ s.t. $|f(z_0)| \geq |f(z)|$, $\forall z \in \Delta$, then f is constant.

ii) If $\Delta \subseteq \mathbb{C}$ is a bounded domain, $f \in \mathcal{H}(\Delta) \cap C(\overline{\Delta})$,

then $\max_{z \in \Delta} |f(z)| = \max_{z \in \partial\Delta} |f(z)|$

Corollary 4:

Let $\Delta \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(\Delta)$.

If $\exists z_0 \in \Delta$ s.t. i) $\operatorname{Re} f(z_0) \geq \operatorname{Re} f(z)$, $\forall z \in \Delta$

OR

ii) $\operatorname{Re} f(z_0) \leq \operatorname{Re} f(z)$, $\forall z \in \Delta$

, then f is constant.

Proof.

Hint: $g_{\pm}(z) = e^{\pm f(z)}$, $z \in \Delta$

Laurent series

Def. 1.

A Laurent series around $z_0 \in \mathbb{C}$ has the form:

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\dots + \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0}}_{\text{the principle part}} +$$

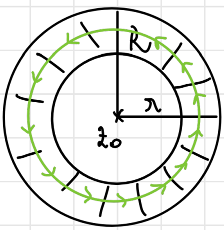
$$+ \underbrace{a_0 + a_n(z-z_0) + \dots + a_n(z-z_0)^n + \dots}_{\text{the Taylor part}},$$

where $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$, w.r.t. the variable $z \in \mathbb{C} \setminus \{z_0\}$.

Theorem 4: (Laurent expansion theorem)

Let $z_0 \in \mathbb{C}$, $0 \le r < R \le \infty$ and $f \in \mathcal{H}(U(z_0, r, R))$.

Then: $\exists!$ Laurent series around z_0 that converges uniformly on compacts of $U(z_0, r, R)$ (both parts, principal and Taylor, converge unif. on comp.)



s.t. $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, $z \in U(z_0, r, R)$,

where $a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$, $\forall \rho \in (r, R)$, $n \in \mathbb{Z}$,

$\gamma_\rho(t) = z_0 + \rho \cdot e^{it}$, $t \in [0, 2\pi]$

Isolated singular points

Def. 2.

Let $f \in \mathcal{H}(G)$, $G \subseteq \mathbb{C}$ is open.

$z_0 \in \mathbb{C} \setminus G$ is an isolated singular point of f,

if $\exists\, r > 0$ s.t. $\dot{D}(z_0, r) \subseteq G$.

Moreover, we say that z_0 is

i) removable, if $\exists \lim\limits_{z \to z_0} f(z) = \infty \in \mathbb{C}$ (the limit at z_0 is finite)

ii) pole, if $\exists \lim\limits_{z \to z_0} f(z) = \infty$ (the limit at z_0 exists in $\mathbb{C}_\infty$ and is ∞)

iii) essential, if $\nexists \lim\limits_{z \to \infty} f(z)$ (the limit at z_0 does NOT exist in $\mathbb{C}_\infty$)

Characterization of isolated singular points

Let $z_0 \in \mathbb{C}$, $r > 0$ and $f \in \mathcal{H}(\dot{D}(z_0, r))$.

Consider the Laurent series expansion of f

around z_0: $(*)$ $f(z_0) = \dots + a_{-n}$