Lecture 1

introduction to differential equations

2h lectures	Seminary Test	2p.
2h seminar	Laboratory Test	1p
1h laboratory	Written exam	7p

$\left.\right\}$ 10p.

Bibliography

1. S.L. Campbell, R. Haberman, Introduction to diff. eq with dynamical systems, Princeton Univ. Press, 2008.
2. M.A. Şerban, Ecuaţii şi sisteme de ecuaţii diferenţiale, Presa Univ. 2009.
3. Gh. Micula, P. Pavel, Ecuaţii diferenţiale şi integrale prin probleme şi exerciţii, Ed. Dacia, 1989.

Laboratory software : Maple

S. Lynch, Dynamical systems with applications using Maple, Birkhauser, 2001

1. Equations and solutions

$x^2 - x = 0$ $\quad$ x - unknown

$x(x-1) = 0$ $\quad$ $x \in \mathbb{R}$, $x \in \mathbb{Z}$

$x_1 = 0$, $x_2 = 1$

Differential equation
unknown is a function $y = y(x)$

Example 1

$y'(x) = y(x)$

$y(x) = e^x$ is a solution
$y(x) = 0$ is a solution
$y(x) = c \cdot e^x$, $c \in \mathbb{R}$.

$$y'(x) = (c e^x)' = c \cdot (e^x)' = c \cdot e^x = y(x)$$

$\boxed{y(x) = c \cdot e^x, \ c \in \mathbb{R}}$ — the general solution of the equation

Example 2

$y'(x) = f(x)$, $f \in C(I)$ given function

the general solution

$y(x) = \int f(x)\,dx + c$, $c \in \mathbb{R}$

or

$y(x) = \int_{x_0}^{x} f(s)\,ds + c$, $c \in \mathbb{R}$, where $x_0 \in I$.

Definition

By a diff. eq. we understand an equation which has as an unknown a function and in its expression appears the derivatives of the unknown function.

General form : $\underbrace{\boxed{F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0}}_{\text{the implicit form of the diff. eq.}}$ (1)

x - independent variable

y - unknown function

n - the order of the diff. eq.

(2) $\boxed{y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))}$ the explicit form of a diff. eq.

(the normal form or Cauchy form)

$f : D_f \to \mathbb{R}$, $D_f \subseteq \mathbb{R}^n$

D_f - is called the domain of the diff. eq.

Examples

a) $y' + y^2 = x^2$ - first order diff. eq.

$y' = -y^2 + x^2$ - the normal form

b) $y^{(4)} \cdot y'' + y' \cdot y = 0$ - a forth order diff. eq.

c) $y''' \cdot y + y'' \cdot \cos(y) + x^2 = 0$ - third order diff. eq.

Definition

A function $y \in C^n(I)$ is a solution of the diff. eq (2) if:

(i) $I \subseteq \mathbb{R}$ is an interval;

(ii) $(x, y(x), y'(x), \dots, y^{(n-1)}(x)) \in D_f$, $\forall x \in I$;

(iii) $y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$, $\forall x \in I$.

2. First order diff. equation

$n = 1$

$$\boxed{y'(x) = f(x, y(x))} \quad (3)$$

$$f : D_f \to \mathbb{R} \quad , D_f \subseteq \mathbb{R}^2$$

Definition

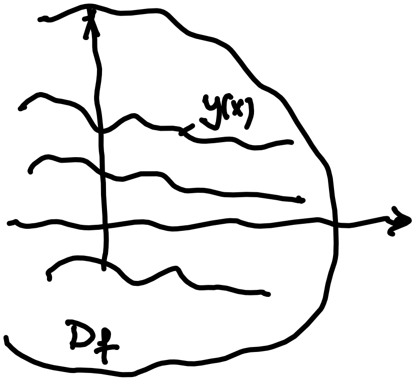
A function $y \in C^1(I)$ is a solution of the diff. eq. (3) if:

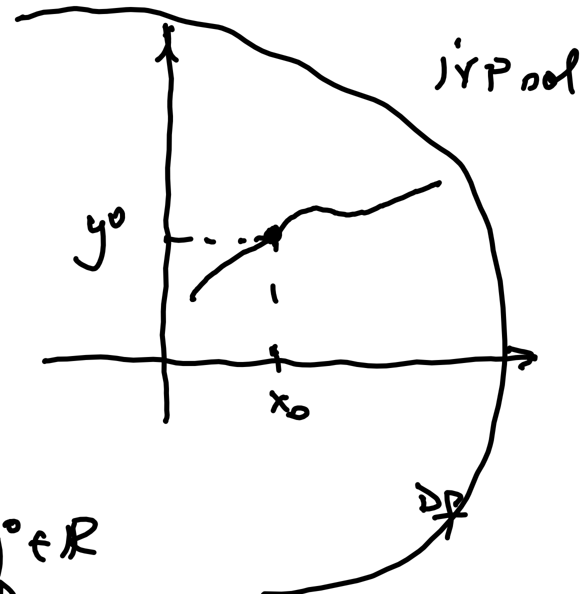
(i) $I \subseteq \mathbb{R}$ is an interval;

→ (ii) $(x, y(x)) \in D_f$, $\forall x \in I$;

(iii) $y'(x) = f(x, y(x))$, $\forall x \in I$.

(ii) $(x, y(x)) \in D_f$, $\forall x \in I \iff G_y = \{(x, y(x)) : x \in I\} \subseteq D_f$.





initial value problem (IVP)

$$(4)\quad \begin{cases} y' = f(x,y) \\ y(x_0) = y^0 \end{cases} \qquad x_0 \in I,\ y^0 \in \mathbb{R} \quad (x_0, y^0) \in D_f.$$

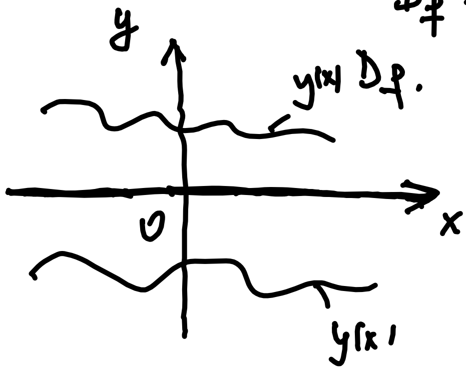
If the (IVP) (4) has an unique solution then $(x_0, y^0) \in D_f$ is called an existence and uniqueness point.

Otherwise, the point (x_0, y^0) is called a singular point

Examples

1) $y' = -\frac{x}{y}$ $f(x,y) = -\frac{x}{y}$

$$D_f = \mathbb{R} \times \mathbb{R}^* = \underbrace{\mathbb{R} \times (-\infty, 0)}_{U_1} \cup \underbrace{\mathbb{R} \times (0, +\infty)}_{U_2}.$$



The solution graph is contained in U_1 or in U_2

$$y' = -\frac{x}{y} \Rightarrow y \cdot y' = -x \,/\cdot 2$$

$$\boxed{2y \cdot y' = -2x} \Longleftrightarrow \boxed{2 \cdot y(x) \cdot y'(x) = -2x}$$

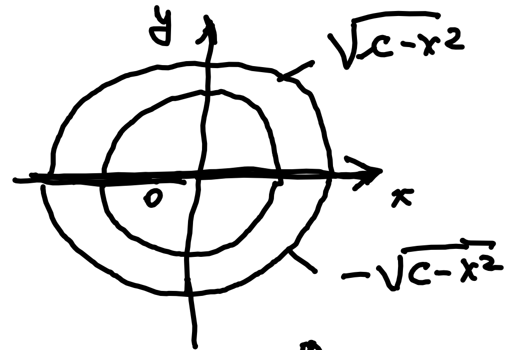
$$(y^2)' = -2x$$

$$y^2 = \int (-2x)\,dx + c$$

$$\boxed{y^2 = -x^2 + c \,,\, c \in \mathbb{R}}$$

$$\boxed{x^2 + y^2 = c \,,\, c \in \mathbb{R}}$$ ← the general sol. in implicit.

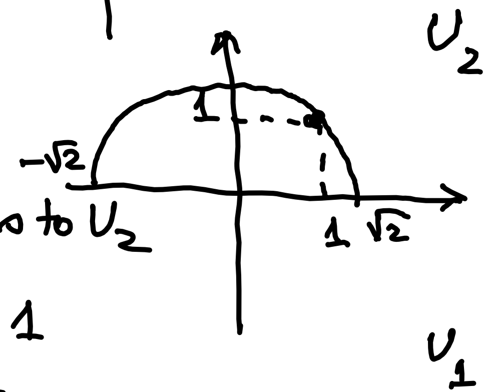
$$\boxed{y(x) = \pm\sqrt{c - x^2}, \quad c \in \mathbb{R}}$$



Consider the following (IVP)

$$\begin{cases} y' = -\dfrac{x}{y} \\ y(1) = 1 \end{cases}$$

$x_0 = 1$, $y^0 = 1$, $(x_0, y^0) = (1,1) \in U_2$



$y(x) = \sqrt{c - x^2} \longrightarrow$ the graph belongs to U_2

$y(1) = 1 \Rightarrow \sqrt{c-1} = 1 \Rightarrow c - 1 = 1$

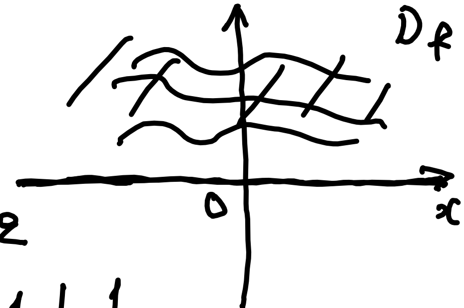
$c = 2$

the (IVP) solution is $\boxed{y(x) = \sqrt{2 - x^2}}$ $\quad y : \underbrace{(-\sqrt{2}, \sqrt{2})}_{I} \to \mathbb{R}$

$(1,1)$ is an existence and uniqueness point.

2) $y' = \sqrt{y}$ $\qquad f(x,y) = \sqrt{y}$

$D_f = \mathbb{R} \times [0, +\infty)$



$\boxed{y(x) \equiv 0}$ is a solution of the diff. eq

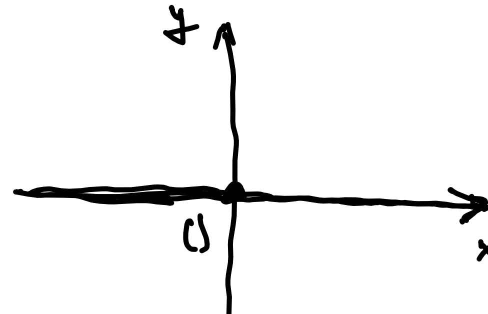
$y \neq 0$

$$y' = \sqrt{y} \quad \Big| : \sqrt{y} \quad \Rightarrow \quad \frac{y'}{\sqrt{y}} = 1 \quad \Big| \cdot \frac{1}{2}$$

$$\Rightarrow \underbrace{\frac{y'}{2\cdot\sqrt{y}}}_{(\sqrt{y})'} = \frac{1}{2} \Rightarrow (\sqrt{y})' = \frac{1}{2} \Rightarrow \sqrt{y} = \int \frac{1}{2}\,dx + c$$

$$\sqrt{y} = \frac{1}{2}x + c, \quad c \in \mathbb{R}$$

$$\Rightarrow \boxed{y(x) = \left(\frac{1}{2}x + c\right)^2, \ c \in \mathbb{R}}$$ the general sol.



Consider the ivP

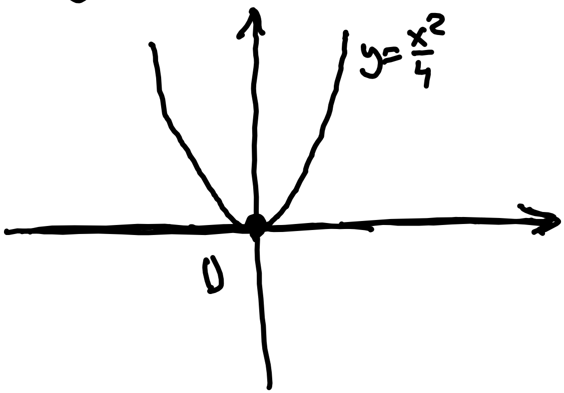
$$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases} \qquad x_0 = 0,\ y^0 = 0$$

$y(x) \equiv 0$ is a solution of the ivP

$y(x) = \left(\frac{1}{2}x + c\right)^2$

$y(0) = 0 \Rightarrow c = 0 \Rightarrow \boxed{y(x) = \frac{x^2}{4}}$ is a solution of IVP

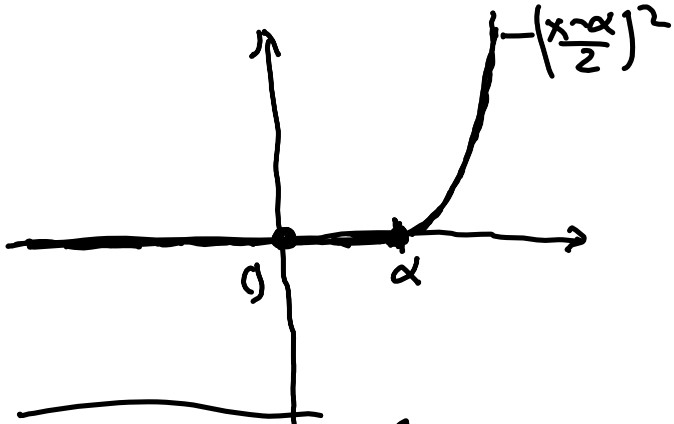
$(0,0)$ is a singular point



$\alpha > 0$

$$y_\alpha(x) = \begin{cases} 0, & x \le \alpha \\ \left(\frac{x-\alpha}{2}\right)^2, & x > \alpha. \end{cases}$$

y_α is a solution of IVP for $\forall \alpha > 0$



$\boxed{y(x) = \left(\frac{1}{2}x + c\right)^2}$

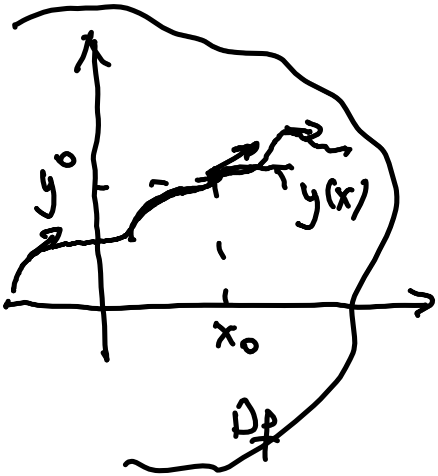
$x = \alpha \Rightarrow y(\alpha) = 0$

$\left(\frac{1}{2}\alpha + c\right)^2 = 0 \Rightarrow c = -\frac{1}{2}\alpha.$

$y(x) = \left(\frac{x-\alpha}{2}\right)^2$

3. Geometrical interpretation

What does a diff. eq $y' = f(x,y)$ tell us geometrically?



$(x_0, y^0) \in D_f$ we can evaluate

$f(x_0, y^0) \longrightarrow$ the slope of y in the point (x_0, y^0)

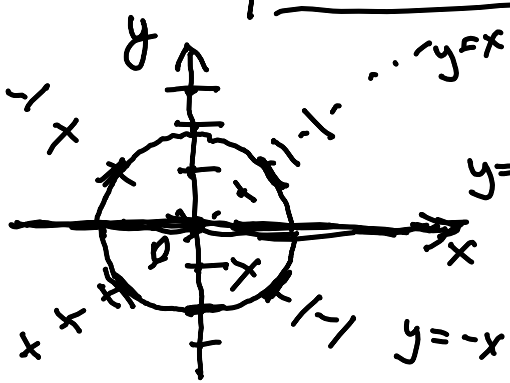
$y'(x_0) = f(x_0, y^0)$

Let's consider the eq:

$$\boxed{y' = -\frac{x}{y}}$$

$f(x,y) = -\frac{x}{y}$

$D_f = \mathbb{R} \times \mathbb{R}^*$



$x = 0 \Rightarrow f(0,y) = 0 \quad \forall y \in \mathbb{R}^*$

$y = x: \; f(x,y) = f(x,x) = -1$

$y = -x: \; f(x,y) = f(x,-x) = 1$

Solving a diff. equation means to find a function for which the slope in every point is given.

4. Systems of differential equations

$y_1(x), y_2(x), \dots, y_n(x)$ — are unknown functions.

First order diff. eq. system

$$\begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ \vdots \\ y_n' = f_n(x, y_1, \dots, y_n) \end{cases}$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad Y' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \quad f : D_f \to \mathbb{R}^n, \quad D_f \subseteq \mathbb{R}^{n+1}$$

$$(5) \quad \boxed{Y' = f(x, Y)}$$

the vectorial form of a first order diff. eq. syst.

Definition

A function $\underline{y} \in C^1(I, \mathbb{R}^n)$ is a solution of the system (5) if:

(i) $I \subseteq \mathbb{R}$ an interval;

(ii) $(x, \underline{y}(x)) \in D_f$, $\forall x \in I$;

(iii) $\underline{y}'(x) = f(x, \underline{y}(x))$, $\forall x \in I$.