Lecture 2

Analysis in Banach spaces. Contraction principle

$X \neq \emptyset \quad f: X \to X$

Problem. Find $x \in X$ such that $\boxed{f(x) = x}$ the fixed point equation

a solution $x^* \in X$ such that $f(x^*) = x^*$ is called a fixed point of f.

$$F_f = \text{Fix}(f) = \{x \in X \mid f(x) = x\}$$ the set of fixed points

Def. $(X, +, \cdot, K)$ is called a vector (linear space) over the field K iff.:

(i) $(X, +)$ abelian group

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$, $\forall x \in X, \alpha, \beta \in K$

(iii) $\alpha(x+y) = \alpha x + \alpha y$, $\forall x, y \in X, \forall \alpha \in K$.

(iv) $\exists 1 \in K$ s.t $1 \cdot x = x$, $\forall x \in X$.

Examples

$K = \mathbb{R}$ or $K = \mathbb{C}$

1) $(\mathbb{R}^n, +, \cdot, \mathbb{R})$

$$x + y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad x, y \in \mathbb{R}^n$$

$\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$

$$\lambda x = \lambda \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

$(\mathbb{R}^n, +, \cdot, \mathbb{R})$ is a real linear space

2) $C[a,b] = \{ x : [a,b] \to \mathbb{R} \mid x \text{ is cont. on } [a,b] \}$

$(C[a,b], +, \cdot, \mathbb{R})$

$x, y \in C[a,b]$ $\quad x + y \quad (x+y)(t) = x(t) + y(t)$

$\lambda \in \mathbb{R}, x \in C[a,b]$ $\quad \lambda \cdot x \quad (\lambda \cdot x)(t) = \lambda \cdot x(t)$

linear space over $\mathbb{R}$.

3) $C([a,b],\mathbb{R}^n) = \{ x : [a,b] \to \mathbb{R}^n \mid x \text{ cont on } [a,b] \}$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$x+y$: $\quad x, y \in C([a,b],\mathbb{R}^n)$

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

$$(x+y)(t) = \begin{pmatrix} x_1(t) + y_1(t) \\ \vdots \\ x_n(t) + y_n(t) \end{pmatrix}$$

$\lambda \cdot x$: $\quad (\lambda x)(t) = \begin{pmatrix} \lambda x_1(t) \\ \vdots \\ \lambda x_n(t) \end{pmatrix}$

$(C([a,b],\mathbb{R}^n), +, \cdot, \mathbb{R})$ is a real linear space.

Def. Let $(X, +, \cdot, \mathbb{R})$ be a real linear space.
A functional $p: X \to \mathbb{R}$ is called a norm if:

(i) $p(x) \geq 0$, $\forall x \in X$ and $p(x) = 0 \iff x = 0_X$ (positivity)

(ii) $p(\lambda x) = |\lambda| \cdot p(x)$, $\forall x \in X, \forall \lambda \in \mathbb{R}$ (homogeneity)

(iii) $p(x+y) \leq p(x) + p(y)$, $\forall x, y \in X$ (triangle ineq.).

usualy $p(x) \overset{not}{=} \|x\|$

If $\|\cdot\|$ is a norm on X then $(X, \|\cdot\|)$ is called a normed space.

Examples

1) $\mathbb{R}$, $|\cdot|$ $\Rightarrow$ $(\mathbb{R}, |\cdot|)$ is a normed space.

$x \in \mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

2) $\mathbb{R}^n$, $\|x\|_E = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$ - the Euclidian norm

$\|x\|_C = \max_{i=\overline{1,n}} |x_i|$ - the Chebyshev norm

$\|x\|_M = \sum_{i=1}^{n} |x_i|$ - Minkowski norm

Def. $\|\cdot\|_1, \|\cdot\|_2$ two norms on X

The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are strongly equivalent if.

$\exists\, c_1, c_2 > 0$ s.t $c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \cdot \|x\|_1, \forall x \in X.$

$\|\cdot\|_1, \|\cdot\|_2$ are strongly equiv $\overset{\text{not}}{\Longleftrightarrow}$ $\|\cdot\|_1 \sim \|\cdot\|_2$

Remark: $\|\cdot\|_E \sim \|\cdot\|_C \sim \|\cdot\|_M$ in $\mathbb{R}^n$

3) $X = C[a,b]$

$\|x\|_C = \max\limits_{t \in [a,b]} |x(t)|$ – Chebyshev norm

$\|x\|_B = \max\limits_{t \in [a,b]} \left(|x(t)| \cdot e^{-\tau(t-a)} \right), \tau > 0$ –

– Bielecki norm

$\|\cdot\|_C \sim \|\cdot\|_B$

4) $X = C([a,b], \mathbb{R}^n)$

$$\|x\|_C = \max_{i=\overline{1,n}} \max_{t\in[a,b]} |x_i(t)| \quad \text{– Chebyshev norm}$$

$$\|x\|_B = \max_{i=\overline{1,n}} \max_{t\in[a,b]} \left(|x_i(t)| \cdot e^{-\tau(t-a)} \right) \quad \text{– Bielecki norm}, \quad \tau > 0$$

$$\|\cdot\|_C \sim \|\cdot\|_B$$

Def. Let $(X, +, \cdot, \mathbb{R}, \|\cdot\|)$ be a normed space.

(i) A sequence $(x_n) \subseteq X$ is convergent to $x^* \in X \Leftrightarrow$
$\Leftrightarrow \forall \varepsilon > 0 \; \exists n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$ we have $\|x_n - x^*\| < \varepsilon$. $\left(\|x_n - x^*\| \xrightarrow[n\to\infty]{} 0 \right)$

(ii) A sequence $(x_n) \subseteq X$ is a fundamental seq. (Cauchy seq.)
$\Leftrightarrow \forall \varepsilon > 0 \; \exists n_\varepsilon \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$ we have $\|x_n - x_m\| < \varepsilon$

$$\left(\|x_n - x_m\| \xrightarrow[\substack{n\to\infty \\ m\to\infty}]{} 0 \right)$$

Remark

Any convergent sequence is a Cauchy sequence

Reciprocally is not always true.

$(\mathbb{Q}, |\cdot|)$

$x_n = \left(1+\frac{1}{n}\right)^n$, $(x_n) \subseteq \mathbb{Q}$

- $x_n \longrightarrow e \notin \mathbb{Q}$

(x_n) is a Cauchy seq., but is not convergent in $\mathbb{Q}$.

if x_n converges to $x^* \in X \overset{not}{\Longleftrightarrow}$

$\Longleftrightarrow \lim\limits_{n\to\infty} x_n = x^*$ or $x_n \longrightarrow x^*$

$x_n \xrightarrow[\|\cdot\|]{} x^*$

Def. If in a normed space any Cauchy sequence is convergent the space is called a complete space

Any normed and complete spaces are called Banach spaces.

Examples

$(\mathbb{R}, |\cdot|)$, $(\mathbb{R}^n, \|\cdot\|)$
$\|\cdot\|_E$, $\|\cdot\|_c$, $\|\cdot\|_M$

$(C[a,b], \|\cdot\|)$
$\|\cdot\|_c$, $\|\cdot\|_B$
$\|\cdot\|_c \sim \|\cdot\|_B$

$(C([a,b], \mathbb{R}^n), \|\cdot\|)$
$\|\cdot\|_c$, $\|\cdot\|_B$

Banach spaces.

Operators on normed spaces

$$f : (X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$$

Def.
a) f is a continuous operator if $\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0$ s.t if $\|x - y\|_X < \delta$ then $\|f(x) - f(y)\|_Y < \varepsilon \Leftrightarrow$
$\Leftrightarrow$ if for every $(x_n) \subseteq X$, $x_n \xrightarrow[X]{} x^*$ then $f(x_n) \xrightarrow[Y]{} f(x^*)$

b) f is α-lipschitz if $\exists \alpha > 0$ such that:

$$\|f(x) - f(y)\|_Y \leq \alpha \cdot \|x - y\|_X, \quad \forall x, y \in X.$$

α is called the lipschitz constant.

if $\alpha < 1$ then f is called α-contraction.

<u>Remark</u>. Any lipschitz operator is a continuous operator.

<u>Theorem</u> (contraction principle - Banach principle) (1922)

Let $(X, \|\cdot\|)$ be a Banach space, $f : X \to X$ is an α-contraction. Then:

(a) $F_f = \{x^*\}$.

(b) the succesive approximation sequence $(f^n(x_0))_{n \in \mathbb{N}}$ converges to x^* for every $x_0 \in X$.

(c) $$\|f^n(x_0) - x^*\| \leq \frac{\alpha^n}{1 - \alpha} \cdot \|f(x_0) - x_0\|, \quad \forall x_0 \in X.$$

The succesive approximation sequence (Picard sequence)

$x_0 \in X$ starting point

$$x_1 = f(x_0)$$
$$x_2 = f(x_1) = f(f(x_0)) = (f\circ f)(x_0) \overset{not}{=} f^2(x_0)$$
$$\vdots$$
$$x_n = f(x_{n-1}) = (\underbrace{f\circ\ldots\circ f}_{n \text{ times}})(x_0) \overset{not}{=} f^n(x_0)$$

<u>Proof</u>. $x_0 \in X$ $\quad \boxed{x_n = f(x_{n-1})} \iff x_n = f^n(x_0)$

the Picard sequence

(x_n) is a Cauchy sequence

$$\| x_{n+1} - x_n \| = \| f(x_n) - f(x_{n-1}) \| \leq \alpha \cdot \| \underset{f(x_{n-1})}{x_n} - \underset{f(x_{n-2})}{x_{n-1}} \| \leq$$

$$\leq \alpha^2 \cdot \| x_{n-1} - x_{n-2} \| \leq \ldots \leq \alpha^n \cdot \| x_1 - x_0 \|$$

$$\boxed{\| x_{n+1} - x_n \| \leq \alpha^n \| x_1 - x_0 \|}$$

$$\boxed{(x_n) \text{ is a Cauchy sequence} \iff \|x_{n+p} - x_n\| \xrightarrow[n\to+\infty]{} 0, \ \forall p \in \mathbb{N}.}$$

$$\underline{\|x_{n+p} - x_n\|} \leq \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \ldots + \|x_{n+1} - x_n\|$$

$$\leq \alpha^{n+p-1} \cdot \|x_1 - x_0\| + \alpha^{n+p-2} \cdot \|x_1 - x_0\| + \ldots + \alpha^n \|\cdot x_1 - x_0\|$$

$$= \alpha^n \cdot \|x_1 - x_0\| \left(\alpha^{p-1} + \alpha^{p-2} + \ldots + 1\right)$$

$$= \alpha^n \cdot \|x_1 - x_0\| \cdot \frac{1-\alpha^p}{1-\alpha} \leq \underset{\downarrow \atop 0}{\alpha^n} \cdot \|x_1 - x_0\| \cdot \frac{1}{1-\alpha}$$

$$\Rightarrow \|x_{n+p} - x_n\| \xrightarrow[n\to\infty]{} 0, \ \forall p \in \mathbb{N}$$

$$\left.\begin{array}{l} \Rightarrow (x_n) \text{ is a Cauchy seq.} \\ X \text{ is Banach space} \end{array}\right\} \Rightarrow \exists x^* \in X \text{ s.t. } x_n \to x^*$$

$$\left.\begin{array}{l} x_{n+1} = f(x_n) \\ n \to \infty \\ f \text{ is cont.} \end{array}\right\} \Rightarrow x^* = f(x^*) \Rightarrow x^* \in F_f .$$

x^* is a unique fixed point of f.

suppose $x^*, y^* \in F_f \Rightarrow$

$$\Rightarrow \|x^* - y^*\| = \|f(x^*) - f(y^*)\| \leq \alpha . \|x^* - y^*\|$$

$$\Rightarrow \underbrace{(1-\alpha)}_{>0} \|x^* - y^*\| \leq 0 \Rightarrow 0 \leq \|x^* - y^*\| \leq 0$$

$$\Rightarrow \|x^* - y^*\| = 0 \Rightarrow x^* = y^* .$$

(c) is obtained from:

$$\|x_{n+p} - x_n\| \leq \frac{\alpha^n}{1-\alpha} (1-\alpha^p) . \|x_1 - x_0\|$$

$$p \to +\infty \quad \|x^* - x_n\| \leq \frac{\alpha^n}{1-\alpha} \|x_1 - x_0\| .$$

Theorem. (The abstract date dependence th.)

Let $(X, \|\cdot\|)$ be Banach space, $f, g : X \to X$

We suppose that:

(i) f is α-contraction ($F_f = \{x_f^*\}$)

(ii) $F_g \neq \emptyset$ ($x_g^* \in F_g$)

(iii) $\exists \eta > 0$ s.t. $\|f(x) - g(x)\| \leq \eta$, $\forall x \in X$.

Then: $\|x_f^* - x_g^*\| \leq \dfrac{\eta}{1-\alpha}$, $\forall x_g^* \in F_g$

Proof: $\|x_f^* - x_g^*\| = \|f(x_f^*) - g(x_g^*)\| \leq$

$\leq \|f(x_f^*) - f(x_g^*)\| + \|f(x_g^*) - g(x_g^*)\|$

$\leq \alpha \cdot \|x_f^* - x_g^*\| + \eta$

$\Rightarrow \|x_f^* - x_g^*\| \leq \alpha \cdot \|x_f^* - x_g^*\| + \eta \Rightarrow$

$(1-\alpha)\|x_f^* - x_g^*\| \leq \eta \Rightarrow \|x_f^* - x_g^*\| \leq \dfrac{\eta}{1-\alpha}$