Lecture 3

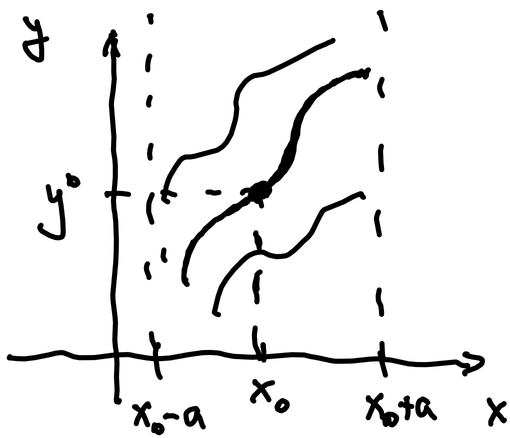
The Cauchy Problem (initial Value Problem)

The Existence and Uniqueness Theorems.

(1) $\begin{cases} y' = f(x,y) \\ y(x_0) = y^0 \end{cases}$ (2)

$f : [x_0 - a, x_0 + a] \times \mathbb{R} \to \mathbb{R}$, $a > 0$

$y^0 \in \mathbb{R}$.



Lemma. The problem (1)+(2) $\Leftrightarrow$ (3)

(3) $y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds$

(Volterra integral equation)

$$\boxed{A(y)(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds.}$$

$y \in C[x_0 - a, x_0 + a]$

$y \longmapsto A(y)$

(3) $\Leftrightarrow$ $y(x) = A(y)(x)$, $\forall x \in [x_0 - a, x_0 + a]$ $\Leftrightarrow$

$\Leftrightarrow \quad y = A(y)$

y is a solution of (1)+(2) $\Leftrightarrow$ y is a solution of (3) $\Leftrightarrow$

$\Leftrightarrow \quad y \in F_A$ (y is a fixed point of operator A).

$\vdots \quad \boxed{f \in C([x_0-a, x_0+a] \times \mathbb{R}, \mathbb{R})} \Rightarrow$

$\Rightarrow$ if $y \in C[x_0-a, x_0+a]$ then $A(y) \in C[x_0-a, x_0+a]$

$$A : C[x_0-a, x_0+a] \to C[x_0-a, x_0+a]$$

$\|\cdot\|_C$ – Cebishev norm

$$\|y\|_C = \max_{x \in [x_0-a, x_0+a]} |y(x)|$$

$\Rightarrow (C[x_0-a, x_0+a], \|\cdot\|_C)$ is a Banach space.

$\|\cdot\|_B$ – Bielecki norm

$$\boxed{\|y\|_B = \max_{x \in [x_0-a, x_0+a]} \left(|y(x)| \cdot e^{-\tau |x-x_0|} \right)}, \quad \tau > 0$$

$(C[x_0-a, x_0+a], \|\cdot\|_B)$ is a Banach space.

The Contraction Principle

$(X, \|\cdot\|)$ be a Banach space and $A: X \to X$ be an α-contraction, i.e.

$$\|A(y) - A(z)\| \le \alpha \cdot \|y - z\|, \quad \forall y, z \in X,$$

then:

(i) $F_A = \{y^*\}$.

(ii) any succesive approximation sequence $A^n(y_0)$ convergece to the unique fixed point y^* for all starting point $y_0 \in X$

(iii) $\| A^n(y_0) - y^* \| \le \dfrac{\alpha^n}{1-\alpha} \cdot \|A(y_0) - y_0\|, \ \forall y_0 \in X$.

$$X = (C[x_0 - a, x_0 + a], \|\cdot\|_B)$$

Let $y, z \in C[x_0 - a, x_0 + a]$

$$|A(y)(x) - A(z)(x)| = \left| \cancel{y^0} + \int_{x_0}^{x} f(s, y(s))\,ds - \right.$$

$$\left. - \cancel{y^0} - \int_{x_0}^{x} f(s, z(s))\,ds \right| =$$

$$= \left| \int_{x_0}^{x} f(s, y(s))\,ds - \int_{x_0}^{x} f(s, z(s))\,ds \right| =$$

$$= \left| \int_{x_0}^{x} \big(f(s, y(s)) - f(s, z(s)) \big)\,ds \right| \le$$

$$\le \left| \int_{x_0}^{x} \left| f(s, y(s)) - f(s, z(s)) \right| ds \right| \le$$

...

$$\boxed{\exists L_f > 0 : |f(s,u) - f(s,v)| \le L_f \cdot |u - v|, \quad \forall s \in [x_0 - a, x_0 + a], \ \forall u, v \in \mathbb{R}}$$

$$\le \left| \int_{x_0}^{x} L_f \cdot |y(s) - z(s)|\,ds \right| =$$

$$= \left| \int_{x_0}^{x} L_f \underbrace{|y(s) - z(s)| \cdot e^{-\tau|s - x_0|}}_{\le \|y - z\|_B} e^{\tau|s - x_0|}\,ds \right| \le$$

$$\le \left| \int_{x_0}^{x} L_f \cdot \|y - z\|_B \cdot e^{\tau|s - x_0|}\,ds \right| =$$

$$= L_f \cdot \|y-z\|_B \left| \int_{x_0}^{x} e^{\tau|s-x_0|}\, ds \right| \le L_f \cdot \|y-z\|_B \cdot \frac{1}{\tau} \cdot e^{\tau|x-x_0|}$$

if $x \ge x_0 \Rightarrow x \ge s \ge x_0 \Rightarrow |s-x_0| = s-x_0$

$$\left|\int_{x_0}^{x} e^{\tau|s-x_0|}\, ds\right| = \left|\int_{x_0}^{x} e^{\tau(s-x_0)}\, ds\right| =$$

$$= \left| \frac{1}{\tau} \cdot e^{\tau(s-x_0)} \Big|_{x_0}^{x} \right| = \frac{1}{\tau}\left| \underbrace{e^{\tau(x-x_0)}}_{\ge 1} - 1\right| \le$$

$$\le \frac{1}{\tau} e^{\tau(x-x_0)} = \frac{1}{\tau} e^{\tau|x-x_0|}$$

if $x \le x_0 \Rightarrow x \le s \le x_0 \Rightarrow |s-x_0| = x_0 - s.$

$$\left|\int_{x_0}^{x} e^{\tau|s-x_0|}\, ds\right| = \left|\int_{x_0}^{x} e^{\tau(x_0-s)}\, ds\right| =$$

$$= \left|\left(-\frac{1}{\tau}\right) e^{\tau(x_0-s)} \Big|_{x_0}^{x} \right| = \frac{1}{\tau}\left| e^{\tau(x_0-x)} - 1\right| =$$

$x_0 - x \ge 0 \Rightarrow e^{\tau(x_0-x)} - 1 \ge 0$

$\tau > 0$

$$= \frac{1}{\tau}\left(e^{\tau(x_0-x)} - 1\right) \le \frac{1}{\tau} e^{\tau(x_0-x)} = \frac{1}{\tau} e^{\tau|x-x_0|}$$

$$\Rightarrow |A(y)(x) - A(z)(x)| \le L_f \cdot \|y-z\|_B \frac{1}{\tau} e^{\tau|x-x_0|} \ldots, \quad \forall x \in [x_0-a, x_0+a]$$

$$\Rightarrow |A(y)(x) - A(z)(x)| \cdot e^{-\tau|x-x_0|} \le L_f \frac{1}{\tau} \|y-z\|_B, \quad -\text{''}-$$

$$\Rightarrow \max_{x\in[x_0-a,x_0+a]} |A(y)(x) - A(z)(x)| \cdot e^{-\tau|x-x_0|} \le L_f \cdot \frac{1}{\tau} \|y-z\|_B$$

$$\Rightarrow \quad \|A(y) - A(z)\|_B \le \frac{L_f}{\tau} \cdot \|y-z\|_B, \quad \forall y,z \in C[x_0-a, x_0+a]$$

$\Rightarrow$ A is lipschitz with the constant $L_A = \frac{L_f}{\tau}$

if we choose $\tau = L_f + 1 \Rightarrow L_A = \frac{L_f}{L_f+1} < 1 \Rightarrow$

$\Rightarrow$ A is a contraction

$(C[x_0-a, x_0+a], \|\cdot\|_B)$ Banach space

$\Big\} \Rightarrow$ contraction Principle

$\Rightarrow \quad F_A = \{y^*\}$

$A^n(y_0) \longrightarrow y^*$

the estimation

Theorem (The existence and uniqueness theorem in the space)

Let consider the Cauchy problem (1)+(2)

Suppose that:

(i) $f \in C([x_0-a, x_0+a] \times \mathbb{R}, \mathbb{R})$ (cont. condition)

(ii) f is L_f - lipschitz with respect to the second variable on $\mathbb{R}$, i.e.

$$\exists L_f > 0 \text{ s.t } |f(s,u) - f(s,v)| \le L_f \cdot |u-v|, \quad \forall s \in [x_0-a, x_0+a], \ \forall u,v \in \mathbb{R}.$$

Then

(a) the Cauchy problem (1)+(2) has an unique solution $y^* \in C[x_0-a, x_0+a]$

(b) the unique sol. y^* can be obtained from any successive approximation sequence starting from any point $y_0 \in C[x_0-a, x_0+a]$

(c) we have the estimation:

$$\| A^n(y_0) - y^* \|_B \le \frac{L_A^n}{1-L_A} \cdot \| A(y_0) - y_0 \|_B, \quad \forall y_0 \in C[x_0-a, x_0+a]$$

where $L_A = \frac{L_f}{1+L_f}$

Remark 1) $u \in C^1(\mathbb{R})$

if. $\exists M > 0$ such that $|u'(x)| \leq M$ $\forall x \in \mathbb{R}$, then u is lipschitz with the constant $L_u = M$.

2) $f \in C^1([x_0-a; x_0+a] \times \mathbb{R}, \mathbb{R})$, $f = f(x,y)$

if $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq M$, $\forall x \in [x_0-a, x_0+a]$, $\forall y \in \mathbb{R}$ $\Rightarrow$

$\Rightarrow$ f is lipschitz with respect to the second variable ($L_f = M$)

Example: $f(x,y) = \sqrt{y}$

$f : [-a, a] \times [0, +\infty) \to \mathbb{R}$.

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|\frac{1}{2\sqrt{y}}\right| \xrightarrow[y \to 0]{} +\infty$$

$\Rightarrow$ $\left|\frac{\partial f}{\partial y}(x,y)\right|$ is not bounded $\Rightarrow$ f is not lipschitz with respect to the second variable.

$$\boxed{\begin{array}{l} \begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases} \\ y(x) \equiv 0,\ y(x) = \frac{x^2}{4} \\ \text{are sol. of the IVP} \end{array}}$$

Suppose that we have

(1) $y' = f(x, y)$
(2) $y(x_0) = y^0$

and

(4) $z' = g(x, z)$
(5) $z(x_0) = z^0$

$$\boxed{f, g \in C([x_0 - a, x_0 + a] \times \mathbb{R}, \mathbb{R}), \; y^0, z^0 \in \mathbb{R}.}$$

The Abstract Data Dependence theorem

$(X, \|\cdot\|)$ a Banach space, $A, B : X \to X$ such that

(i) A is L_A contraction ($\{y^*\} = F_A$)

(ii) $\exists z^* \in F_B$

(iii) $\|A(y) - B(y)\| \leq \eta$, $\forall y \in X$

Then: $\|y^* - z^*\| \leq \dfrac{\eta}{1 - L_A}.$

$$(1)+(2) \Leftrightarrow (3) \quad y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$$

$$(4)+(5) \Leftrightarrow (6) \quad z(x) = z^0 + \int_{x_0}^{x} g(s, z(s))\,ds.$$

$$A, B : C[x_0 - a, x_0 + a] \longrightarrow C[x_0 - a, x_0 + a]$$

$$y \in C[x_0 - a, x_0 + a] : A(y)(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$$

$$B(y)(x) = z^0 + \int_{x_0}^{x} g(s, y(s))\,ds.$$

$$|A(y)(x) - B(y)(x)| = \left| y^0 + \int_{x_0}^{x} f(s, y(s))\,ds - z^0 - \int_{x_0}^{x} g(s, y(s))\,ds \right|$$

$$\leq |y^0 - z^0| + \left| \int_{x_0}^{x} \big(f(s, y(s)) - g(s, y(s)) \big)\,ds \right| \leq$$

$$\leq \underbrace{|y^0 - z^0|}_{\leq \eta_1} + \left| \int_{x_0}^{x} \underbrace{|f(s, y(s)) - g(s, y(s))|}_{\leq \eta_2}\,ds \right| \leq$$

$$\leq \eta_1 + \eta_2 \left| \int_{x_0}^{x} ds \right| = \eta_1 + \eta_2 |x - x_0| \leq \eta_1 + \eta_2 \cdot a$$

$$|A(y)(x) - B(y)(x)| \leq \eta_1 + \eta_2 a \;\Big|\cdot e^{-\delta|x - x_0|} \Rightarrow$$

$$|A(y)(x) - B(y)(x)|\, e^{-\tau|x-x_0|} \le (\eta_1+\eta_2 a)\cdot \underbrace{e^{-\tau|x-x_0|}}_{\le 1} \le \eta_1+\eta_2 a$$

$$\Rightarrow \quad \|A(y) - B(y)\|_B \le \eta_1+\eta_2 a$$

<u>Theorem</u> (The Continuous Data Dependence Theorem)

Let us consider the problems (1)+(2) and (4)+(5)

Suppose:

(i) $f, g \in C([x_0-a, x_0+a]\times\mathbb{R}, \mathbb{R})$

(ii) f is L_f - lipschitz with respect to second variable on $\mathbb{R}$.

(iii) $|y^0 - z^0| \le \eta_1$, $|f(x,u) - g(x,u)| \le \eta_2$, $\forall x \in [x_0-a, x_0+a]$, $\forall u \in \mathbb{R}$

Then

$$\|y^* - z^*\|_B \le \frac{\eta}{1-L_A} \quad \text{where } L_A = \frac{L_f}{L_f+1}, \; y^* \text{ is the unique sol. of (1)+(2) and } z^* \text{ is a sol. of (4)+(5).}$$

$$\boxed{\eta = \eta_1 + \eta_2 \cdot a}$$

$$\left.\begin{matrix}\eta_1 \to 0\\ \eta_2 \to 0\end{matrix}\right. \Rightarrow \eta \to 0 \Rightarrow$$