Lecture 4

Initial Value Problem. The Existence and Uniqueness Theorems (II)

(1) $\begin{cases} y' = f(x, y) \\ y(x_0) = y^0 \end{cases}$ (2) $\quad f : [x_0 - a, x_0 + a] \times \mathbb{R} \to \mathbb{R}$, $a > 0$, $y^0 \in \mathbb{R}$.

$(1)+(3) \Longleftrightarrow (3)\quad y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$ Volterra integral equation

Theorem 1 (The Existence and Uniqueness Th. in space)
Let us consider the IVP (1)+(2). Suppose that:

(i) $f \in C([x_0 - a, x_0 + a] \times \mathbb{R}, \mathbb{R})$ (continuity hypothesis)

(ii) f is Lipschitz with respect to the second variable, i.e
$\exists L_f > 0$ s.t. $|f(x, u) - f(x, v)| \le L_f \cdot |u - v|$, $\forall x \in [x_0 - a, x_0 + a]$, $\forall u, v \in \mathbb{R}$

Then:

(a) The IVP (1)+(2) has a unique sol. $y^* \in C([x_0 - a, x_0 + a])$

(b) The unique sol y^* can be obtained from any successive approximation seq. starting from any point $y_0 \in C([x_0 - a, x_0 + a])$

(c) error estimation.

$(3) \Rightarrow \quad A : C[x_0-a, x_0+a] \longrightarrow C[x_0-a, x_0+a]$

$$A(y)(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds .$$

A is a contraction on $C[x_0-a, x_0+a]$ with $L_A = \frac{L_f}{L_f+1}$.

$\left(C[x_0-a, x_0+a], \|\cdot\|_B \right)$

$$\|y\|_B = \max_{x\in[x_0-a, x_0+a]} |y(x)| \cdot e^{-\tau|x-x_0|}, \quad \tau := \frac{L_f}{L_f+1}$$

from Contraction Principle $\Rightarrow A^n(y_0) \xrightarrow{\|\cdot\|} y^*$

$y_0 \in C[x_0-a, x_0+a]$ $\qquad \underline{\underline{y^0 \in \mathbb{R}}}$

succesive approximation sequence starting from y_0

$$\boxed{y_{n+1}(x) = y^0 + \int_{x_0}^{x} f(s, y_n(s))\,ds} \qquad y_n = A^n(y_0)$$

$\Updownarrow$

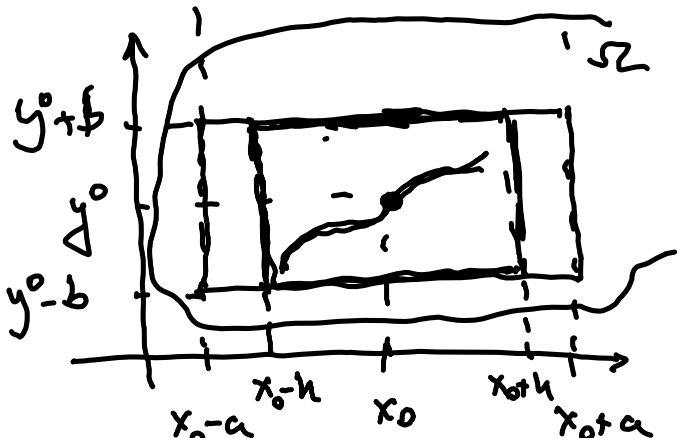
$y_{n+1} = A(y_n)$ $\qquad A^n = \underbrace{A \circ A \circ \ldots \circ A}_{n \text{ times}}.$

Remark: Suppose $f(x,\cdot)$ is C^1 function on $\mathbb{R}$.

f is lipschitz with respect to the second variable $\Longleftrightarrow$

$$\Longleftrightarrow \quad \left|\frac{\partial f}{\partial y}(x,y)\right| \le M, \quad \forall x \in [x_0-a, x_0+a], \ \forall y \in \mathbb{R}$$

$L_f = M$. $\qquad f : [x_0-a, x_0+a] \times \mathbb{R} \to \mathbb{R}$



Suppose that

$f : \Omega \to \mathbb{R}$

$\Omega \subseteq \mathbb{R}^2$, $(x_0, y^0) \in \Omega$

$a, b > 0$

$\Rightarrow \overline{D} = [x_0-a, x_0+a] \times [y^0-b, y^0+b]$

...

$$X = \left(C([x_0-a, x_0+a]), \|\cdot\|_C \right)$$

$$\|y\|_C = \max_{[x_0-a, x_0+a]} |y(x)|$$

$$\overline{B}(y^0, b) = \{ y \in C[x_0-a, x_0+a] \mid \|y - y^0\|_C \le b \} \subseteq X$$

the closed ball centered in y^0 with radius b.

$\left.\begin{array}{l}\overline{B}(y^0, b) \text{ is closed subset of } X. \\ X \text{ is a Banach space}\end{array}\right\} \Rightarrow (\overline{B}(y^0, b), \|\cdot\|_c)$ is a Banach space.

$(1)+(2) \Leftrightarrow (3) \quad y(x) = \underbrace{y^0 + \int_{x_0}^{x} f(s, y(s))\,ds}_{A(y)(x)}.$

$A : X \to X.$

A is well defined if $\boxed{f \in C(\Omega, \mathbb{R}) \text{ (continuity hypothesis)}}$.

$A : \overline{B}(y^0, b) \longrightarrow X$

$A : \overline{B}(y^0, b) \xrightarrow{?} \overline{B}(y^0, b)$

$y \in \overline{B}(y^0, b) \overset{?}{\Longrightarrow} A(y) \in \overline{B}(y^0, b)$?

$A(y) \in \overline{B}(y^0, b) \Leftrightarrow \|A(y) - y^0\|_c \leq b$ ..

Let $y \in \overline{B}(y^0, b) \Rightarrow \|y - y^0\|_c \leq b \Leftrightarrow$

$\Leftrightarrow \max_{x \in [x_0 - a, x_0 + a]} |y(x) - y^0| \leq b \Leftrightarrow$

$|y(x)-y^0| \leq b, \forall x \in [x_0-a, x_0+a]$

$$|A(y)(x)-y^0| = \left|\int_{x_0}^{x} f(s,y(s))\,ds\right| \leq \left|\int_{x_0}^{x} |f(s,y(s))|\,ds\right| \leq \bullet$$

$$\left(\begin{array}{l} f\in C(\Omega,\mathbb{R}) \Rightarrow \left.\begin{array}{l} f\in C(\overline{D},\mathbb{R}) \\ \overline{D} \text{ is a bounded set} \end{array}\right\} \Rightarrow \\ \Rightarrow \exists M_f>0 \text{ s.t. } |f(x,u)| \leq M_f, \forall (x,u)\in\overline{D} \end{array} \right)$$

$$\bullet \leq \left|\int_{x_0}^{x} M_f\,ds\right| = M_f\cdot\left|\int_{x_0}^{x} ds\right| = M_f|x-x_0| \leq M_f\cdot a$$

$$|A(y)(x)-y^0| \leq M_f\cdot a, \forall x\in[x_0-a, x_0+a]$$

$$\left.\begin{array}{l} \underset{\max_x}{\Rightarrow} \|A(y)-y^0\|_C \leq M_f\cdot a \leq b \\ \text{suppose } \boxed{M_f\cdot a \leq b} \end{array}\right\} \Rightarrow A(y)\in\overline{B}(y^0,b)$$

$\Rightarrow A : \overline{B}(y^0, b) \longrightarrow \overline{B}(y^0, b)$

using the same technique from the proof of Existence and Uniqueness Th. from the space $\Rightarrow$

A is contraction with $L_A = \dfrac{L_f}{L_f+1}$ $\left(L_A = \dfrac{L_f}{6} \right)$

we need to suppose that f is lipschitz with the constant L_f on $\overline{D}$, i.e. $\exists L_f > 0$

- $\boxed{|f(x,u) - f(x,v)| \le L_f \cdot |u-v|, \ \forall (x,u), (x,v) \in \overline{D}.}$

Theorem 2. (The Existence and Uniqueness Th. in the ball).

Let consider the IVP (1)+(2), where $f : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^2$ domain (open and conex set). Suppose that:

(i) $f \in C(\Omega, \mathbb{R})$. (continuity hypothesis)

(ii) let $a, b > 0$, $\overline{D} = [x_0 - a, x_0 - a] \times [y^0 - b, y^0 + b]$

f is lipschitz with respect to the second variable on $\overline{D}$

$\exists L_f > 0$ s.t $|f(x,u) - f(x,v)| \le L_f \cdot |u-v|, \forall (x,u), (x,v) \in \overline{D}$

Then:

(a) the IVP (1)+(2) has a unique solution $y^* \in C([x_0-h, x_0+h], [y^0-b, y^0+b])$

where $h = \min\{a, \frac{b}{M_f}\}$, $M_f = \max_{(x,u)\in D} |f(x,u)|$ ...

$$y^* \in \overline{B}(y^0, b) \iff \|y^* - y^0\|_c \leq b \iff$$

$$\iff |y^*(x) - y^0| \leq b, \ \forall x \in [x_0-h, x_0+h] \iff$$

$$\iff \quad y^*(x) \in [y^0-b, y^0+b].$$

(b) the unique sol. y^* can be obtained from any successive approximation sequence starting from any $y_0 \in \overline{B}(y^0, b)$

$$A^n(y_0) \longrightarrow y^*, \ \forall y_0 \in \overline{B}(y^0, b) \iff$$

$$\iff y_0 \in C([x_0-h, x_0+h], [y^0-b, y^0+b])$$

(c) we have the error estimates from the Existence and Uniqueness Th. in space.

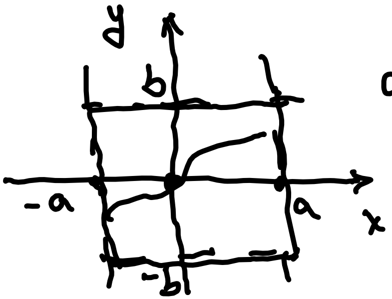
Example

$$\begin{cases} y' = 2x^2 + 3 \cdot y^4 \\ y(0) = 0 \end{cases}$$

$$\boxed{f(x,y) = 2x^2 + 3 \cdot y^4}$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ f is cont on $\mathbb{R}^2$

$x_0 = 0$, $y^0 = 0$ ...



$a > 0$ $\quad$ $f : [-a,a] \times \mathbb{R} \to \mathbb{R}$.

f is cont on $[-a,a] \times \mathbb{R}$.

f is lipschitz with respect to y on $[-a,a] \times \mathbb{R}$?

....

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = |12 \cdot y^3| = 12 \cdot |y|^3 \xrightarrow[|y| \to +\infty]{} +\infty \Rightarrow$$

$\Rightarrow$ f is not lipschitz on $[-a,a] \times \mathbb{R}$

$\Rightarrow$ we cannot apply The E.U.Th. in space.

$a, b > 0$: $\bar{D} = [-a,a] \times [-b,b]$.

$\bar{D} \subset \mathbb{R}^2$ $\Rightarrow$ f cont on $\bar{D}$.

f cont on $\mathbb{R}^2$

$C\left([-h,h],[-b,b]\right)$ where $h=\min\left\{a,\frac{b}{M_f}\right\}$.

$M_f = \max\limits_{\bar{D}} |f(x,y)|$

$|f(x,y)| = |2x^2+3\cdot y^4| \le 2|x^2| + 3\cdot |y|^4 \le 2a^2+3b^4$

$x\in[-a,a]$, $y\in[-b,b]$

$$\Rightarrow \boxed{h=\min\left\{a,\frac{b}{2a^2+3b^4}\right\}.}$$

f is lipschitz with respect to y on $\bar{D}$?

$\left|\frac{\partial f}{\partial y}(x,y)\right| = 12\cdot|y|^3 \le 12\cdot b^3 \Rightarrow f$ is lipschit with respect to y on $\bar{D}$

$\underset{\text{EV.Th. in ball}}{\Longrightarrow}$ $\exists!\, y^*\in\bar{B}(0,b) \Leftrightarrow y^*\in C\left([-h,h],[-b,b]\right)$

where $h=\min\left\{a,\frac{b}{2a^2+3b^4}\right\}$.

for $a=1$, $b=1$. $\Rightarrow h=\min\left\{1,\frac{1}{5}\right\}=\frac{1}{5}$

$y^*\in C\left(\left[-\frac{1}{5},\frac{1}{5}\right],[-1,1]\right)$.

The successive approximation sequence.

$(A^n(y_0)) \Leftrightarrow y_0$ a starting point

$$y_{n+1} = A(y_n)$$

$$A(y)(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds.$$

$\underline{y_0 \in \overline{B}(0,b)} \Leftrightarrow y_0 \in \underline{C([-h,h],[-b,b])}$

successive approximation sequence:

$$y_{n+1}(x) = \int_0^x \left[2s^2 + 3(y_n(s))^4\right] ds$$

$$y_{n+1}(x) = 2 \cdot \frac{s^3}{3}\Big|_0^x + 3 \cdot \int_0^x (y_n(s))^4\,ds.$$

$$\boxed{y_{n+1}(x) = \frac{2}{3} \cdot x^3 + 3 \cdot \int_0^x (y_n(s))^4\,ds}$$

if $y_0 \in C([-h,h],[-b,b]) \Rightarrow (y_n) \longrightarrow y^*$

usually $y_0(x) \equiv y^0 \in \overline{B}(y^0, b)$, $\underline{\forall b > 0}$

for our example, we can choose

$$y_0(x) \equiv 0 \Rightarrow y_1(x) = \frac{2}{3}x^3 + \int_0^x (\underbrace{y_0(s)}_{=0})^4 ds = \frac{2}{3}x^3$$

$$y_2(x) = \frac{2}{3}x^3 + \int_0^x y_1(s) ds = \frac{2}{3}x^3 + \int_0^x \frac{2}{3}s^3 ds =$$

$$= \frac{2}{3}x^3 + \frac{2}{12} \cdot s^4 \Big|_0^x = \frac{2}{3}x^3 + \frac{1}{6}x^4$$

$\vdots$

$$y_n(x) \simeq y^*(x)$$