Lecture 5

Mathematical models given by first order differential equation

1) Radioactive decay

The Rutherford Law: The desintegration rate of a radioactive substance is proportional with the quantity of the substance present at that time.

$R(t) \rightarrow$ the quantity of the subst. at the moment $t>0$.

$R_0 \rightarrow$ the initial quantity of the substance at the initial moment $t=0$

$\boxed{R(0)=R_0}$ the initial condition.

$t \longrightarrow t+\Delta t$

$R(t) \qquad R(t+\Delta t)$

the change rate $\dfrac{R(t+\Delta t)-R(t)}{\Delta t} \xrightarrow[\Delta t\to 0]{} R'(t)$

$R' \propto R \implies \begin{cases} R' = -k \cdot R, & \underline{k>0} \\ R(0) = R_0 \end{cases}$

$R(t)$ is a decreasing function $\to R'(t) < 0$

k - the disintegration constant

$R' = -k \cdot R$ a homogeneous first order linear eq.

$$\frac{dR}{dt} = -k.R \implies \int \frac{dR}{R} = \int -k\,dt \implies \ln R = -k \cdot t + \ln c$$

$$\to \boxed{R(t) = c \cdot e^{-k \cdot t}, \ c \in \mathbb{R}}$$ the gen. sol. of. diff. eq.

$R(0) = R_0 \implies c = R_0$

$\implies$ the model solution

$$\boxed{R(t) = R_0 \cdot e^{-k \cdot t}}$$

$T_{1/2}$ – the half-life time of a radioactive substance is the length of the time that it takes to decay to half of its original size.

$$\underbrace{\begin{array}{l} t=0 \longrightarrow R_0 \\ t=T_{1/2} \longrightarrow \frac{R_0}{2} \end{array}} \Rightarrow R(T_{1/2}) = \frac{R_0}{2}$$

$$R_0\, e^{-kT_{1/2}} = \frac{R_0}{2} \quad \Big| : R_0$$

$$\Rightarrow e^{-kT_{1/2}} = \frac{1}{2} \Rightarrow -k \cdot T_{1/2} = \ln\left(\frac{1}{2}\right) \Rightarrow$$

$$2^{-1}$$

$$\Rightarrow -k \cdot T_{1/2} = -\ln(2) \quad \Big| \cdot (-1)$$

$$k \cdot T_{1/2} = \ln(2)$$

$$\Downarrow$$

$$\boxed{k = \frac{\ln 2}{T_{1/2}}} \quad \text{or} \quad \boxed{T_{1/2} = \frac{\ln 2}{k}}.$$

$$R(t) = R_0 \cdot e^{-kt} =$$

$$= R_0 \cdot e^{-\frac{\ln 2}{T_{1/2}} \cdot t} = R_0 \cdot \left(e^{\ln 2}\right)^{-\frac{t}{T_{1/2}}} = R_0 \cdot 2^{-\frac{t}{T_{1/2}}}$$

$$\boxed{R(t) = R_0 \cdot 2^{-\frac{t}{T_{1/2}}}}$$

2) <u>Radiocarbon Dating</u> (Willard Libby 1950, in 1960 Nobel Prize)

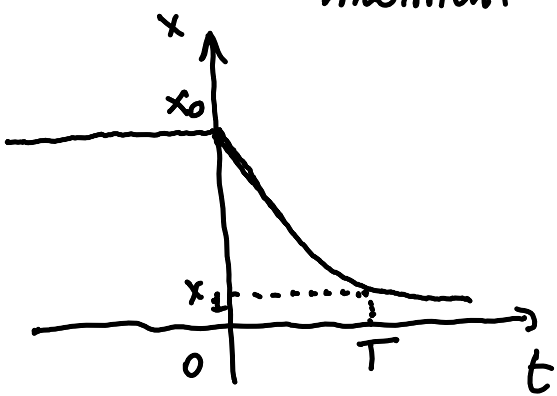
- the method finds an approximating age of some fossilized matter.
- the theory of radiocarbon dating is based on the fact that radioisotop C^{14} is produced in the atmosphere by the action of cosmic radiation.

C^{14} is a radioactive substance with the half-life time $T_{1/2} \simeq 5730$ years $\Rightarrow$ $\boxed{k = \frac{\ln 2}{5730} \text{ years}^{-1}}$

- the ratio of the C^{14} to the stable isotope C^{12} in the amtmosphere appears to be constant and as a consequence the amount of C^{14} present in the living body is also constant (the same as in the atmosphere).
- when the living body dies, the absorbtion of C^{14} ceases and the existing amount of C^{14} decays.

$x(t)$ — the amount of C^{14}/C^{12} at the moment $t>0$

x_0 — the initial amount of C^{14}/C^{12} at the moment $t=0$



$t=0$ is the moment when the body dies.

$$k = \frac{\ln 2}{5730}$$

$$\begin{cases} x' = -kx \\ x(0) = x_0 \end{cases}$$

$$\boxed{x(t) = x_0 \cdot e^{-kt}}$$

...

at the moment $T>0$ it is measured the amount of C^{14}/C^{12} from the remains $\Rightarrow x_1$

$$x(T) = x_1 \Rightarrow \boxed{x_0 \cdot e^{-kT} = x_1}$$

$$\Rightarrow e^{-kT} = \frac{x_1}{x_0} \Rightarrow -k.T = \ln \underbrace{\frac{x_1}{x_0}}_{<1} \Rightarrow k.T = -\ln \frac{x_1}{x_0}.$$

$$\Rightarrow k.T = \ln \frac{x_0}{x_1} \Rightarrow \boxed{T = \frac{1}{k} \cdot \ln \frac{x_0}{x_1}}$$

3) The Thermal Cooling

The Cooling Newton's Law:

The rate of change of the surface temperature of an object is proportional to the difference between the object temperature and the temperature of its surrounding (called the ambient temperature) at the same time.

$T(t)$ – the object temperature at the moment $t>0$

T_0 – the initial object temperature at the initial moment $t=0$.

$\Rightarrow$ $\boxed{T(0)=T_0}$ the initial condition

T_A – the ambiental temperature. (constant value)

$T'(t) \propto \underline{T(t)-T_A}$

$$\begin{cases} T'(t) = (-k)\cdot(T(t)-T_A), & \underline{k>0} \\ T(0)=T_0 \end{cases}$$

Why $k>0$?

if $T(t) < T_A \Rightarrow T(t)$ increase $\Rightarrow T(t)\nearrow \Rightarrow T'(t)>0$

$$\underbrace{T'}_{>0} = \underbrace{-k}_{<0}\cdot \underbrace{[T-T_A]}_{<0}$$

if $T(t) > T_A \Rightarrow T(t)$ is decreasing $\Rightarrow T' < 0$

$$\underbrace{T'}_{<0} = \underbrace{-k}_{<0} \cdot \underbrace{(T - T_A)}_{>0}$$

$T' = -k\,(T - T_A)$ $\qquad T = T(t)$.

separable <u>diff. eq.</u> or $T' + k \cdot T = k \cdot T_A$ nonhomogeneous linear diff. eq.

$$\frac{dT}{dt} = -k \cdot (T - T_A) \Rightarrow$$

$$\Rightarrow \int \frac{dT}{T - T_A} = \int -k \cdot dt \Rightarrow \ln (T - T_A) = -k \cdot t + \ln c$$

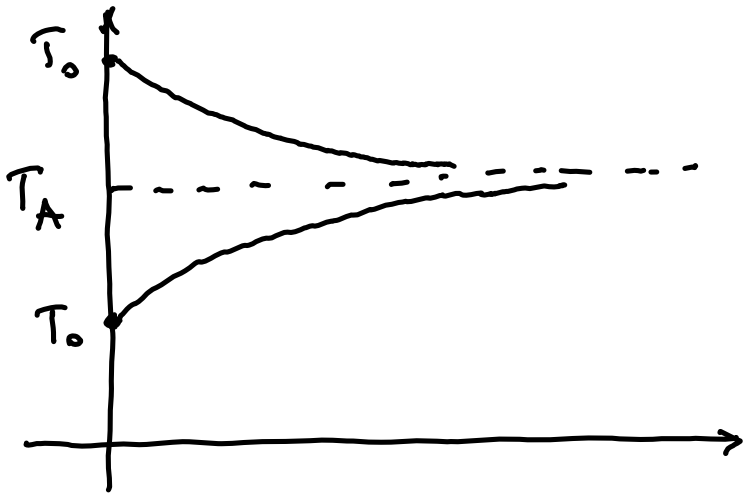
$$\Rightarrow T - T_A = c \cdot e^{-kt} \Rightarrow \boxed{T(t) = T_A + c \cdot e^{-kt},\ c \in \mathbb{R}}$$

the gen. sol.

$T(0) = T_0 \Rightarrow T_A + c = T_0 \Rightarrow c = T_0 - T_A.$

$\Rightarrow$ the model solution:

$$\boxed{T(t) = (T_0 - T_A)\cdot e^{-kt} + T_A}$$



$T(t) \xrightarrow[t\to+\infty]{} T_A, \quad T_0 \neq T_A$

when $T_0 = T_A \Rightarrow$

$\Rightarrow T(t) \equiv T_A.$

4) Population Models for single species

4.1. Exponential growth model (Malthus, 1798)

$N(t)$ — population size at the mom. $t>0$.

N_0 — the initial population size at the initial mom. $t=0$.

$\boxed{N(0)=N_0}$ initial condition

Malthus suppose that per capita growth rate is constant in time.

$\dfrac{N'(t)}{N(t)}$ — the per capita growth rate.

$$\frac{N'}{N}=r$$

$r=b-\mu$

b — the per capita birth rate

μ — the per capita mortality rate.

$\Rightarrow \begin{cases} N'(t) = r \cdot N(t) \\ N(0) = N_0 \end{cases}$ the Malthus model

$N' = rN$ separable eq.

$\frac{dN}{dt} = rN \Rightarrow \int \frac{dN}{N} = \int r \, dt \Rightarrow \ln N = r \cdot t + \ln c$

$\Rightarrow \quad N(t) = c \cdot e^{rt}, \; c \in \mathbb{R}.$

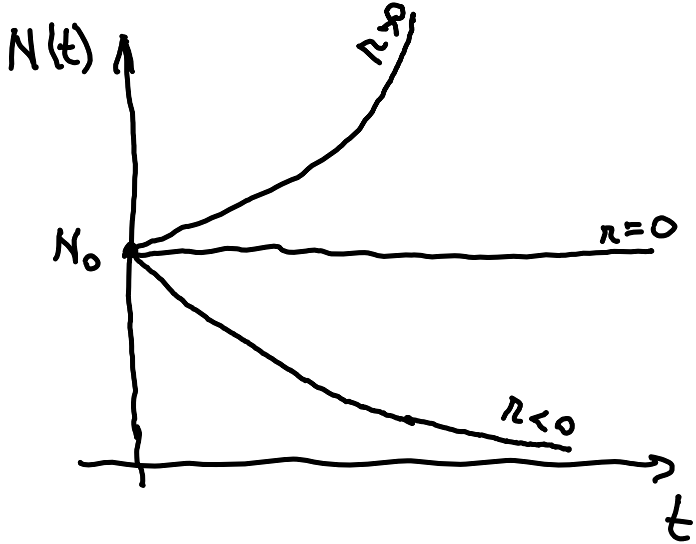
$N(0) = N_0 \Rightarrow c = N_0$

$\Rightarrow$ the model solution: $\boxed{N(t) = N_0 \cdot e^{rt}}$

if $r < 0 \Leftrightarrow b < \mu$ then $N(t) \xrightarrow[t \to +\infty]{} 0$, the population disappears in time.

if $r > 0 \Leftrightarrow b > \mu$ then $N(t) \xrightarrow[t \to \infty]{} +\infty$, the pop. will grow exponentially

if $r = 0 \Leftrightarrow b = \mu$ then $N(t) \equiv N_0$, the pop remain const. in time


N(t)
N0
r>0
r=0
r<0
t