Lecture 6

Population models for Single Species

1) Exponential growth (Malthus 1798)

$N(t)$ – the population size at the moment $t>0$

N_0 – initial population size at the initial moment $t=0$.

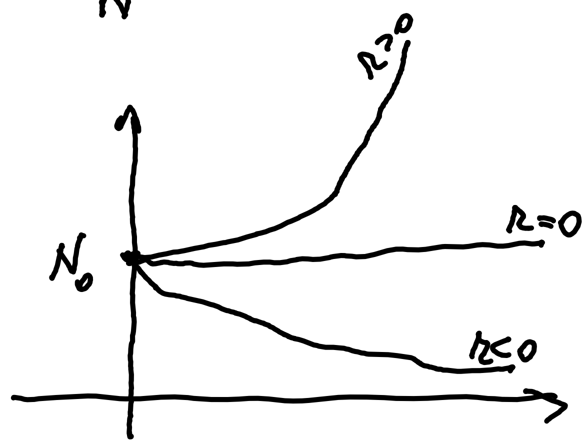
$$N(0)=N_0$$

$\frac{N'}{N}$ – per capita growth rate.

Malthus supposed that $\frac{N'}{N} = r = \text{const}$

$$\rightarrow \begin{cases} N' = r \cdot N \\ N(0) = N_0 \end{cases}$$

$$N(t) = N_0 \cdot e^{rt}$$



2) <u>The logistic model</u> (Verhulst (1838), Pearl-Reed 1920)

- the environmental factors and competion for the resources limit the population growth

- the per capita growth rate depends on population size.

$$r = r(N) \Rightarrow \boxed{\frac{N'}{N} = r(N).}$$

$r(N) = ?$

- per capita growth rate decreases when population size increases.

$\Rightarrow r(N)$ must be a decreasing function

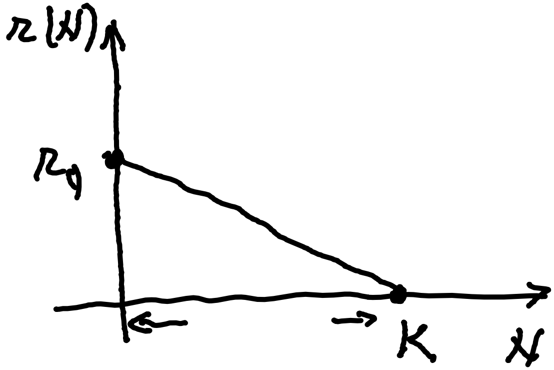
- K - the carrying capacity constant =

= the maximum pop. that environment can support.

$$N(t) \to K \Rightarrow r(N) \to 0$$

- when the population size is small (with respect to K) then the competition phenomenon can be neglected, so in this case the population grows according to the Malthus model

$$N(t) \to 0 \quad \text{then} \quad r(N) \to r_0 \quad \text{intrinsic growth rate.}$$



the function $r(N)$ should interpolate the points $(0, r_0)$ and $(K, 0)$

$$r(N) = r_0\left(1 - \frac{N}{K}\right)$$

$$\Rightarrow \begin{cases} N' = r_0 N\left(1 - \frac{N}{K}\right) \\ N(0) = N_0 \end{cases}$$ the logistic model

$$N' = r_0 \cdot N \cdot \underbrace{\frac{K-N}{K}}$$

$N \equiv 0$, $N \equiv K$ are singular solutions.

$$\frac{dN}{dt} = r_0 \cdot N \cdot \underbrace{\frac{K-N}{K}} \Rightarrow \int \frac{K}{N\cdot(K-N)}\,dN = \int r_0 \cdot dt .$$

$$\frac{K}{N\cdot(K-N)} = \frac{A}{N} + \frac{B}{N-K} \Rightarrow \boxed{K = A\cdot(K-N) + B\cdot N .}$$

$N=0 \Rightarrow K = A\cdot(+K) \Rightarrow A = +1$

$N=K \Rightarrow K = B\cdot K \Rightarrow B = 1$

$$\boxed{\frac{K}{N(K-N)} = \frac{1}{N} + \frac{1}{K-N}}$$

$$\Rightarrow \int\left(\frac{1}{N} + \frac{1}{K-N}\right)dN = \int r_0\,dt \Rightarrow \underline{\ln N} - \underline{\ln|K-N|} = r_0 t + \ln c$$

$$\frac{1}{K-N}$$

$$\Rightarrow \ln\frac{N}{K-N} = r_0 t + \ln c$$

$$\Rightarrow \frac{N}{K-N} = c\cdot e^{r_0 t}$$

$$\Rightarrow N = c\cdot e^{r_0 t}\cdot K - c e^{r_0 t}\cdot N \Rightarrow$$

$$\Rightarrow N\left(1 + c\cdot e^{r_0 t}\right) = c\cdot e^{r_0 t}\cdot K \Rightarrow$$

$$\Rightarrow \boxed{N(t) = \frac{K.c.e^{r_0 t}}{1+c.e^{r_0 t}}, \quad c \in \mathbb{R}} \quad \text{the gen. sol.}$$

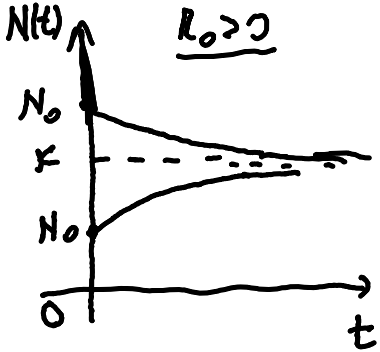
$$N(0) = N_0 \Rightarrow \frac{K.c}{1+c} = N_0 \Rightarrow K.c = N_0 + c N_0$$

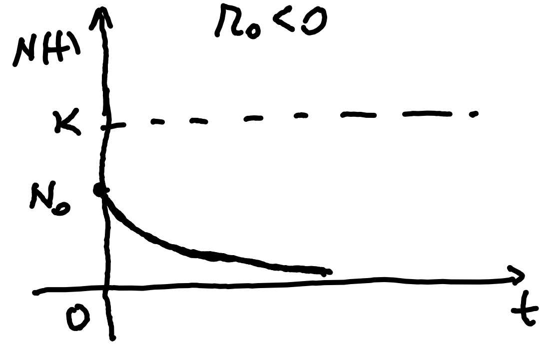
$$\Rightarrow c(K - N_0) = N_0 \rightarrow c = \frac{N_0}{K-N_0}$$

$$\Rightarrow N(t) = \frac{K.\frac{N_0}{K-N_0}.e^{r_0 t}}{1+\frac{N_0}{K-N_0}.e^{r_0 t}} = \frac{K N_0 . e^{r_0 t}}{K - N_0 + N_0 e^{r_0 t}}$$

$$\boxed{N(t) = \frac{K N_0 e^{r_0 t}}{K - N_0 + N_0 e^{r_0 t}} = \frac{K N_0}{(K-N_0)e^{-r_0 t} + N_0}}$$ the logistic model solution.

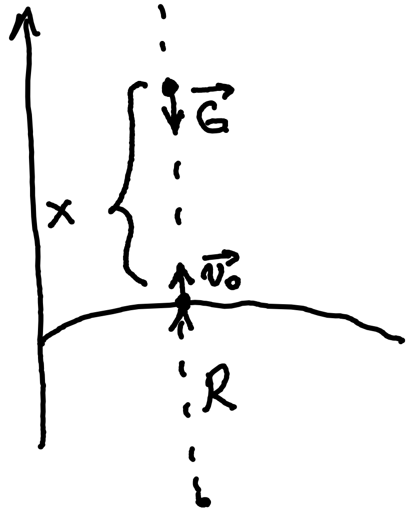
- if. $r_0 > 0 \Rightarrow N(t) \xrightarrow[t \to +\infty]{} K$ the population will tend to the carrying const. capacity
- if $r_0 = 0 \Rightarrow N(t) = N_0$ the pop. remain const in time.
- if $r_0 < 0$ and $\underline{N_0 < K}$. $\Rightarrow N(t) \xrightarrow[t \to +\infty]{} 0$ the pop. will deseappear in time.





3) The vertical throwing. Escape velocity

Problem: A body of a constant mass is projected away from the earth in a direction perpendicular to the earth surface with the initial velocity v_0. Assuming that there is no air resistance but takeing into consideration the variation of the earth's gravitational field, find the expression of the velocity with respect to the distance from the earth's surface.



x – the distance from the body to the earth's surface.

$N(x) = ?$

Law: The gravitational force acting on a body is inversely proportional to the square of the distance from the body to the earth's center.

$$G(x) = -\frac{k}{(x+R)^2}$$

„–" singn means that $G(x)$ is directed in the negative x direction

at the earth' surface

$$x = 0 \Rightarrow G(0) = -m \cdot g \Rightarrow -\frac{k}{R^2} = -mg \Rightarrow \boxed{k = mgR^2}$$

$$\Rightarrow G(x) = -\frac{mgR^2}{(x+R)^2} = -mg \cdot \left(\frac{R}{x+R}\right)^2$$

Newton law : $\boxed{m.a = F}$ $F = G$.

$x = x(t)$ – the body position at the moment t

$\Rightarrow v(t) = x'(t)$

$a(t) = v'(t) = x''(t)$

$\boxed{m.x''(t) = G.}$

$\cancel{m}x''(t) = -\frac{\cancel{m}gR^2}{(x+R)^2}$

$a(t) = v'(t) = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v'(x).x'(t) = v'(x).v.$

$\Rightarrow v'(x).v = -\frac{gR^2}{(x+R)^2}$

$x=0 \quad v(0) = v_0$

$$\Rightarrow \begin{cases} v'(x).v(x) = -\frac{gR^2}{(x+R)^2} \\ v(0) = v_0 \end{cases}$$

$v'(x).v(x) = -\frac{gR^2}{(x+R)^2}$

$\frac{dv}{dx}.v = -\frac{gR^2}{(x+R)^2}$

$v.dv = -\frac{gR^2}{(x+R)^2}.dx \;\Big|\cdot 2$

$\int 2.v.dv = \int -\frac{2gR^2}{(x+R)^2}dx$

$\boxed{v^2 = \frac{2gR^2}{(x+R)} + c,\ c \in \mathbb{R}}$

the gen. sol.

$v(0) = v_0 \Rightarrow x = 0 \quad v = v_0$

$$\Rightarrow v_0^2 = \frac{2gR^{\cancel{2}}}{\cancel{R}} + c \quad \rightarrow c = v_0^2 - 2gR.$$

$\rightarrow$ the model solution:

$$v^2 = \frac{2gR^2}{x+R} + v_0^2 - 2gR$$

$$\Rightarrow \boxed{v(x) = \pm\sqrt{\frac{2gR^2}{x+R} + v_0^2 - 2gR}}$$

"+" — the body is rising

"−" — the body is falling

Maximal altitude

h – maximal altitude $\Rightarrow \boxed{v(h) = 0} \Longrightarrow$

given an initial velocity $v_0 \Rightarrow \underline{h(v_0)} = ?$

$$\Longrightarrow \frac{2gR^2}{h+R} + v_0^2 - 2gR = 0$$

$$h+R = \frac{2gR^2}{2gR - v_0^2} \Rightarrow h = \frac{2gR^2}{2gR - v_0^2} - \overset{2gR - v_0^2}{R} \Rightarrow$$

$$\Rightarrow h = \frac{\cancel{2gR^2} - 2gR^2 + Rv_0^2}{2gR - v_0^2}$$

$$\Rightarrow \boxed{h(v_0) = \frac{Rv_0^2}{2gR - v_0^2}}.$$

Escape velocity

v_e - escape velocity = v_0 such that the body will not return to earth' surface

$$\Rightarrow h \to +\infty$$

$$v_e = \lim_{h \to +\infty} v_0(h)$$

$$\Rightarrow v_e = \lim_{h \to +\infty} \sqrt{\frac{2gRh}{h+R}}$$

$$= \sqrt{2gR} \cong 11.1 \ \frac{Km}{s.}$$

$$h = \frac{Rv_0^2}{2gR - v_0^2}$$

$$2gRh - v_0^2 h = Rv_0^2$$

$$v_0^2 (h+R) = 2gRh$$

$$v_0^2 = \frac{2gRh}{h+R} \Rightarrow v_0(h) = \sqrt{\frac{2gRh}{h+R}}$$