# Lecture 7

## n-order linear differential equations

General form

(1) $y^{(n)} + a_1(x) \cdot y^{(n-1)} + a_2(x) \cdot y^{(n-2)} + \dots + a_{n-1}(x) \cdot y' + a_n(x) \cdot y = f(x)$

$a_1, \dots, a_n, f \in \underline{C(I)}$

$f(x) \not\equiv 0 \Rightarrow$ nonhomogeneous linear diff. eq.

$f(x) \equiv 0 \Rightarrow$ homogeneous linear diff. eq.

**Theorem 1.** The IVP:

$$\begin{cases} y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_n \end{cases} \qquad x_0 \in I, \; y_1, \dots, y_n \in \mathbb{R}$$

has a unique solution in $C^n(I)$.

$L : C^n(I) \to C(I)$

$y \mapsto Ly$

$Ly(x) = y^{(n)}(x) + a_1(x) \cdot y^{(n-1)}(x) + \ldots + a_{n-1}(x) \cdot y'(x) + a_n(x) \cdot y(x)$

operator $L$ is a linear operator.

$u, v \in C^n(I)$, $\alpha, \beta \in \mathbb{R}$

$L(\alpha u + \beta v) = \alpha \cdot Lu + \beta \cdot Lv$

$(1) \Leftrightarrow \boxed{Ly = f}$.

<u>the solution set</u> $S'$ of the eq. (1):

$$\boxed{S' = \ker L + \{y_p\}}$$

where $\ker L = \{ y \in C^n(I) : Ly = 0 \}$

$y_p$ is a particular solution of the nonhomogeneous eq. (1) $(Ly = f)$

## The homogeneous case

$$(2) \quad y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_{n-1} \cdot y' + a_n \cdot y = 0$$

$$a_1, \dots, a_n \in C(I)$$

$S_0$ – the solution set of the eq. (2)

$S_0 = \ker L$

**Theorem 2.** $S_0$ is a linear subspace of the linear space $C^n(I)$ with $\dim S_0 = n$.

**Proof.** $S_0$ is a linear subspace of $C^n(I) \iff$

$\iff u, v \in S_0, \ \alpha, \beta \in \mathbb{R} \Rightarrow \alpha u + \beta v \in S_0$

$$u \in S_0 \Rightarrow u^{(n)} + a_1 . u^{(n-1)} + \dots + a_n u = 0 \ / \cdot \alpha$$

$$v \in S_0 \Rightarrow v^{(n)} + a_1 v^{(n-1)} + \dots + a_n v = 0 \ / \cdot \beta$$

$$(+)$$

$\alpha u^{(n)} + \beta v^{(n)} = (\alpha . u + \beta v)^{(n)}$

$$(\alpha u + \beta v)^{(n)} + a_1 (\alpha u + \beta v)^{(n-1)} + \dots + a_n (\alpha u + \beta v) = 0$$

$$\Rightarrow \alpha u + \beta v \in S_0$$

$\dim S_0 \overset{?}{=} n$

$$\varphi : \mathbb{R}^n \to S_0 \qquad\qquad \alpha \in \mathbb{R}^n \Leftrightarrow \alpha = (\alpha_1, \dots, \alpha_n)$$
$$\alpha \mapsto y(\alpha)$$

where $y(\alpha)$ is the solution of the ivp

$$\begin{cases} Ly = 0 \\ y(x_0) = \alpha_1 \\ y'(x_0) = \alpha_2 \\ \vdots \\ y^{(n-1)}(x_0) = \alpha_n \end{cases} \underset{\text{Th. 1}}{\Longrightarrow} \exists!\, y(\alpha) : I \to \mathbb{R} \text{ sol. of ivp.} \Rightarrow$$

$\Rightarrow \varphi$ is a bijection.

$\varphi$ is isomorphism of linear spaces $\Leftrightarrow$

$$\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta) \quad , \forall \alpha, \beta \in \mathbb{R}^n$$
$$\varphi(\lambda\alpha) = \lambda \varphi(\alpha). \quad , \forall \alpha \in \mathbb{R}^n, \lambda \in \mathbb{R}.$$

$$\varphi(\alpha+\beta) \overset{?}{=} \varphi(\alpha)+\varphi(\beta)$$
$$y(\alpha+\beta) \overset{?}{=} y(\alpha)+y(\beta)$$

$\alpha=(\alpha_1,\dots,\alpha_n)$
$\beta=(\beta_1,\dots,\beta_n)$ $\in \mathbb{R}^n$

$\underline{y(\alpha+\beta)}$ is the sol. of the IVP

$$\begin{cases} Ly=0 \\ y(x_0)=\alpha_1+\beta_1 \\ \vdots \\ y^{(n-1)}(x_0)=\alpha_n+\beta_n \end{cases} \quad (*)$$

...

$y(\alpha)$ is the sol. of the IVP

$$\begin{cases} Ly=0 \\ y(x_0)=\alpha_1 \\ \vdots \\ y^{(n-1)}(x_0)=\alpha_n \end{cases} \quad (**)$$

$y(\beta)$ is the sol. of the IVP:

$$\begin{cases} Ly=0 \\ y(x_0)=\beta_1 \\ \vdots \\ y^{(n-1)}(x_0)=\beta_n \end{cases} \quad (***)$$

$y(\alpha), y(\beta) \in S_0 \Rightarrow$

$\rightarrow y(\alpha)+y(\beta) \in S_0$

$(y(\alpha)+y(\beta))(x_0)=\alpha_1+\beta_1$

$\vdots$

$(y(\alpha)+y(\beta))^{(n-1)}(x_0)=$

$=y^{(n-1)}(\alpha)(x_0)+y^{(n-1)}(\beta)(x_0)=\alpha_n+\beta_n$

$\Rightarrow y(\alpha)+y(\beta)$ is a sol. of the IVP $(*)$ $\rightarrow$

from the $\exists!$ Th. 1 $\Rightarrow$

$\Rightarrow y(\alpha+\beta)=y(\alpha)+y(\beta)$.

analogue $\varphi(\lambda\alpha) = \lambda \cdot \varphi(\alpha)$ $\quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, \lambda \in \mathbb{R}$.

$\downarrow$ sol. of ivp: $\quad \searrow$ sol. of the ivp:

$$\begin{cases} Ly = 0 \\ y(x_0) = \lambda\alpha_1 . \\ \vdots \\ y^{(n-1)}(x_0) = \lambda\alpha_n . \end{cases} \qquad \begin{cases} Ly = 0 \\ y(x_0) = \alpha_1 \\ \vdots \\ y^{(n-1)}(x_0) = \alpha_n \end{cases}$$

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$S_0$ is a linear subspace of $C^n(I)$

$\boxed{\dim S_0 = n} \iff \{y_1, \dots, y_n\} \subset S_0$ basis in $S_0$. $\iff$

$\{y_1, \dots, y_n\}$ is called a fundamental system of solutions.

$\iff \{y_1, \dots, y_n\}$ is linearly independent system.

<u>Def</u>. The system $\{y_1, \dots, y_n\} \subset C(I)$ is

a) linearly dependent iff. $\exists\, c_1, \dots, c_n \in \mathbb{R}$ with $(c_1, \dots, c_n) \neq (0, \dots, 0)$ such that $c_1 y_1 + \dots + c_n y_n = 0$

b) linearly independent iff :

$$c_1 y_1 + \dots + c_n y_n = 0 \implies c_1 = \dots = c_n = 0$$

$$W(x;y_1,\dots,y_n)=\begin{vmatrix} y_1(x) & \dots & y_n(x) \\ y_1'(x) & & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

the wronskian (the determinant of Wronski)

**Theorem 3.**

a) If $\{y_1,\dots,y_n\}\subset C(I)$ is linearly dependent $\Rightarrow$

$\underset{\not\Leftarrow}{\Rightarrow}$ $W(x;y_1,\dots,y_n)\equiv 0$ on $I$.

b) If $\{y_1,\dots,y_n\}\subset S_0$ is linearly independent $\Rightarrow$

$\Rightarrow$ $W(x;y_1,\dots,y_n)\neq 0$, $\forall x\in I$.

**Proof.** a) $\{y_1,\dots,y_n\}$ is linearly dependent $\Rightarrow$

$\Rightarrow \exists\,(c_1,\dots,c_n)\neq(0,\dots,0)$ s.t. $c_1y_1+\dots+c_ny_n=0$

we suppose that $c_1\neq 0$ $\Rightarrow$ $y_1=\frac{1}{c_1}(-c_2y_2-\dots-c_ny_n)$

$y_1$ is a linear combination of $y_2,\dots,y_n$

$y_1' = \frac{1}{c_1}\left(-c_2 y_2' - \ldots - c_n y_n'\right)$ $y_1'$ is the same linear combination of $y_2', \ldots, y_n'$ as in the case of $y_1$

$\vdots$

$y_1^{(n-1)} = \frac{1}{c_1}\left(-c_2 y_2^{(n-1)} - \ldots - c_n y_n^{(n-1)}\right)$

$\Rightarrow$ the first column of $W(x; y_1, \ldots, y_n)$ is a linear combination of the other $n-1$ columns

$\Rightarrow W(x; y_1, \ldots, y_n) = 0, \ \forall x \in I.$

b) We suppose that $\exists\, x_0 \in I$ such that

$W(x_0; y_1, \ldots, y_n) = 0$

We consider the system:

$$(3)\ \begin{cases} c_1 y_1(x_0) + \ldots + c_n y_n(x_0) = 0 \\ \vdots \\ c_1 y_1^{(n-1)}(x_0) + \ldots + c_n y_n^{(n-1)}(x_0) = 0 \end{cases}$$

with unknowns $c_1, \ldots, c_n$

the system (3) is an homogeneous system with the coefficients matrix:

$$A = \begin{pmatrix} y_1(x_0) & \dots & y_n(x_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{pmatrix}$$

$\det A = W(x_0; y_1, \dots, y_n) = 0 \Rightarrow$ the system (3) has at least one solution $(\tilde{c}_1, \dots, \tilde{c}_n) \neq (0, \dots, 0)$.

We construct the function:

$$\left.\begin{matrix} \tilde{y}(x) = \tilde{c}_1 y_1(x) + \dots + \tilde{c}_n y_n(x) \\ y_1, \dots, y_n \in S_0 \end{matrix}\right\} \Rightarrow \tilde{y} \in S_0.$$

$\tilde{y}$ is a solution of the IVP:

$$\begin{cases} Ly = 0 \\ y(x_0) = 0 \\ \vdots \\ y^{(n-1)}(x_0) = 0 \end{cases}$$

but this IVP has as a solution the function $y(x) \equiv 0 \xRightarrow[\text{Th 1}]{} \tilde{y} \equiv 0 \Rightarrow$

$$\left.\begin{matrix} \tilde{c}_1 y_1 + \dots + \tilde{c}_n y_n = 0 \\ \text{with } (\tilde{c}_1, \dots, \tilde{c}_n) \neq (0, \dots, 0) \end{matrix}\right\} \Rightarrow$$

$\Rightarrow \{y_1, \dots, y_n\}$ is a linearly dependent system $\Rightarrow$ contradiction.

<u>Conclusion:</u>
In $S_0$ we have the following possibilities for $y_1, \dots, y_n \in S_0$:

- $\{y_1, \dots, y_n\}$ is linearly dependent $\Rightarrow W(x; y_1, \dots, y_n) \equiv 0$
- $\{y_1, \dots, y_n\}$ is linearly independent $\Rightarrow W(x; y_1, \dots, y_n) \neq 0$ $\forall x \in I$.

<u>Theorem 4</u> (The Wronskian criterion)

The system $\{y_1, \dots, y_n\} \subset S_0$ is a fundamental system of solution for (2) $\Longleftrightarrow$ $\exists x_0 \in I$ such that

$$W(x_0; y_1, \dots, y_n) \neq 0,$$

$\{y_1, \dots, y_n\}$

To solve the equation (2) means to find $\{y_1, \dots, y_n\}$ a fundamental system of solutions $\Longleftrightarrow$

$\Longleftrightarrow$ to find $x_0 \in I$ for which $W(x_0; y_1, \dots, y_n) \neq 0$.

If $\underline{\{y_1, \dots, y_n\}}$ is a <u>fundamental syst.</u> of sol. for (2) then the general solution of (2) is:

$$\boxed{y_0 = c_1 y_1 + \dots + c_n y_n \quad , c_1, \dots, c_n \in \mathbb{R}}$$

## The nonhomogeneous case

(1) $y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f$ $\qquad a_1, \dots, a_n \in C(I)$, $f \in C(I)$.

The general solution of (1) is

$$\boxed{y = y_0 + y_p}$$

where:

$y_0$: is the general solution of the homogeneous eq. (2)

$y_p$: is a particular solution of the nonhomogeneous eq. (1).

$y_p$ can be found using the variation of the constants method.