Lecture 8

Linear differential equations

(1) $y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) \cdot y = f(x)$ $\qquad a_i \in C(I)$, $f \in C(I)$

$f \not\equiv 0$ nonhomogeneous linear eq.

$f \equiv 0$ homogeneous linear eq.

(2) $y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) \cdot y = 0$.

$$L : C^n(I) \to C(I)$$

$$y \mapsto Ly = y^{(n)} + a_1 \cdot y^{(n-1)} + \ldots + a_n \cdot y$$

$\boxed{(1) \Leftrightarrow Ly = f.}$

S – the solution set of (1)

$$\boxed{S = \ker L + \{y_p\}}$$

where $\ker L = \{ y \in C^n(I) \mid Ly = 0 \}$

y_p – a particular sol. of (1).

$\{y_1, \dots, y_n\}$ is a fundamental solution system $\Leftrightarrow$

$\Leftrightarrow$ $y_i \in \ker L$, $i = \overline{1,n}$ and $\{y_1, \dots, y_n\}$ is linear independent system of functions

$\Leftrightarrow$ $y_i \in \ker L$, $i = \overline{1,n}$ and $W(x; y_1, \dots, y_n) \neq 0$

$$W(x; y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & & \vdots \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

the Wronskian.

for a fundamental system of solutions.

$\underline{\{y_1, \dots, y_n\}}$. $\Rightarrow$ $\ker L = \{c_1 y_1 + \dots + c_n y_n \mid c_1, \dots, c_n \in \mathbb{R}\}$.

The general sol. of (1)

$$\boxed{y = y_0 + y_p}$$

$$\boxed{y_0 = c_1 y_1 + \dots + c_n y_n, \ c_1, \dots, c_n \in \mathbb{R}}$$

y_0 – the general sol. of (2)

y_p – a particular sol. of (1). can be found using the variation of the constants method.

Variation of the constants method

$\{y_1, \dots, y_n\}$ a fundamental system of solutions for (2).

$$\Rightarrow y_0 = c_1 y_1 + \dots + c_n y_n \quad , c_1, \dots, c_n \in \mathbb{R}.$$

$y_p = ?$ sol. of (1)

we look for y_p of the form:

$$y_p(x) = c_1(x) \cdot y_1(x) + \dots + c_n(x) \cdot y_n(x).$$

$$y_p'(x) = c_1'(x) \cdot y_1(x) + c_1(x) \cdot y_1' + c_2' \cdot y_2 + c_2 \cdot y_2' + \dots + c_n' \cdot y_n + c_n y_n'$$

we impose the condition that:

$$\boxed{c_1' \cdot y_1 + c_2' y_2 + \dots + c_n' \cdot y_n = 0}.$$

$$\Rightarrow y_p'(x) = c_1 \cdot y_1' + c_2 \cdot y_2' + \dots + c_n \cdot y_n'$$

$$\Rightarrow y_p''(x) = c_1' \cdot y_1' + c_1 \cdot y_1'' + c_2' \cdot y_2' + c_2 \cdot y_2'' + \dots + c_n' \cdot y_n' + c_n \cdot y_n''$$

we impose the condition that:

$$\boxed{c_1' \cdot y_1' + \dots + c_n' \cdot y_n' = 0}$$

$$\Rightarrow y_p''(x) = c_1 \cdot y_1'' + \dots + c_n \cdot y_n''$$

$$\vdots$$

on each step we impose the condition:

$$\boxed{c_1' y_1^{(k)} + \dots + c_n' y_n^{(k)} = 0}, \quad \boxed{k = 0, 1, \dots, n-2.}$$

$$\Rightarrow \boxed{y_p^{(k+1)}(x) = c_1 \cdot y_1^{(k+1)} + \dots + c_n \cdot y_n^{(k+1)}}, \quad k = 0, 1, \dots, n-2.$$

$$k = n-2 \Rightarrow y_p^{(n-1)}(x) = c_1 \cdot y_1^{(n-1)} + \dots + c_n \cdot y_n^{(n-1)}$$

$$y_p^{(n)}(x) = c_1' \cdot y_1^{(n-1)} + c_1 \cdot y_1^{(n)} + \dots + c_n' \cdot y_n^{(n-1)} + c_n \cdot y_n^{(n)}$$

y_p is a sol. of (1):

$$y_p^{(n)} + a_1 \cdot y_p^{(n-1)} + \dots + a_n \cdot y_p = f(x).$$

$$c_1' \cdot y_1^{(n-1)} + c_1 \cdot y_1^{(n)} + \ldots + c_n' \cdot y_n^{(n-1)} + c_n \cdot y_n^{(n)} +$$

$$+ a_1 \left(c_1 \cdot y_1^{(n-1)} + c_2 \cdot y_2^{(n-1)} + \ldots + c_n \cdot y_n^{(n-1)} \right) +$$

$$+ a_2 \left(c_1 \cdot y_1^{(n-2)} + c_2 \cdot y_2^{(n-2)} + \ldots + c_n \cdot y_n^{(n-2)} \right) + \ldots +$$

$$+ a_{n-1} \left(c_1 \cdot y_1' + c_2 \cdot y_2' + \ldots + c_n \cdot y_n' \right) +$$

$$+ a_n \left(c_1 \cdot y_1 + c_2 \, y_2 + \ldots + c_n \cdot y_n \right) = f$$

$$\Rightarrow c_1' \cdot y_1^{(n-1)} + c_2' \, y_2^{(n-1)} + \ldots + c_n' \cdot y_n^{(n-1)} +$$

$$+ c_1 \cdot \Big(\underbrace{y_1^{(n)} + a_1 \cdot y_1^{(n-1)} + \ldots + a_{n-1} \cdot y_1' + a_n \cdot y_1}_{Ly_1 = 0} \Big) + \ldots +$$

$$+ c_n \cdot \Big(\underbrace{y_n^{(n)} + a_1 \cdot y_n^{(n)} + \ldots + a_{n-1} \cdot y_n' + a_n \cdot y_n}_{Ly_n = 0} \Big) = f.$$

$$\Rightarrow \boxed{c_1' \cdot y_1^{(n-1)} + \ldots + c_n' \cdot y_n^{(n-1)} = f.}$$

$$\Rightarrow \quad (3)\begin{cases} c_1' \cdot y_1 + c_2' \cdot y_2 + \dots + c_n' \cdot y_n = 0 \\ c_1' \cdot y_1' + c_2' y_2' + \dots + c_n' \cdot y_n' = 0 \\ \vdots \\ c_1' \cdot y_1^{(n-2)} + c_2' y_2^{(n-2)} + \dots + c_n' \cdot y_n^{(n-2)} = 0 \\ c_1' \cdot y_1^{(n-1)} + c_2' y_2^{(n-1)} + \dots + c_n' y_n^{(n-1)} = f \end{cases}$$

$c_1', \dots, c_n'$ the unknowns.

coefficients matrix of the system (3):

$$A = \begin{pmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}$$

$$\left.\begin{array}{l} \det A = W(x; y_1, \dots, y_n) \\ \{y_1, \dots, y_n\} \text{ is a fundam. system of sol.} \end{array}\right\} \Rightarrow \det A \neq 0 \Rightarrow \text{syst. (3) has an unique solution}$$

$\Rightarrow c_1'(x), \dots, c_n'(x) \xRightarrow{\int} c_1(x), \dots, c_n(x) \Rightarrow y_p(x).$

Linear differential equations with constant coefficients

$a_i(x) \equiv a_i \;, \; i = \overline{1,n}$

(4) $y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_n \cdot y = f \qquad a_1, \dots, a_n \in \mathbb{R}$

(4) is the nonhomogeneous eq.

(5) $y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_n y = 0$

(5) is the homogeneous eq.

The general sol. of (4)

$$\boxed{y = y_0 + y_p}$$

y_0 – the gen. sol. of (5)

y_p – a particular sol. of (4).

$y_0 = ?$

The homogeneous case

(5) $y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_n \cdot y = 0 \qquad a_1, \dots, a_n \in \mathbb{R}$

we try to find solutions of the form:

$$\left.\begin{aligned} y(x) &= e^{r\cdot x} \\ y'(x) &= r \cdot e^{rx} \\ y''(x) &= r^2 \cdot e^{rx} \\ &\vdots \\ y^{(n)}(x) &= r^n \cdot e^{rx} \end{aligned}\right\} \Rightarrow r^n \cdot \cancel{e^{rx}} + a_1 \cdot r^{n-1} \cdot \cancel{e^{rx}} + \dots + a_{n-1} \cdot r \cdot \cancel{e^{rx}} + a_n \cancel{e^{rx}} = 0 \;/ : e^{rx}$$

$$\Rightarrow (6) \; \boxed{r^n + a_1 r^{n-1} + \dots + a_{n-1} \cdot r + a_n = 0}$$ the characteristic equation

$P(r) = r^n + a_1 r^{n-1} + \dots + a_{n-1} \cdot r + a_n$ the characteristic polynomial.

1. The case when (6) has n simple real roots ($r_i \neq r_j$)

$r_1, \dots, r_n \in \mathbb{R}$ sol. of (6) $\Rightarrow$

$\Rightarrow y_1(x) = e^{r_1 x}, \dots, y_n(x) = e^{r_n x}$ are sol. of (5).

$$W(x; y_1, \dots, y_n) = \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & & y_n^{(n-1)} \end{vmatrix} =$$

$$= \begin{vmatrix} e^{r_1 x} & \dots & e^{r_n x} \\ r_1 e^{r_1 x} & & r_n \cdot e^{r_n x} \\ \vdots & & \vdots \\ r_1^{n-1} \cdot e^{r_1 x} & & r_n^{n-1} \cdot e^{r_n x} \end{vmatrix} = e^{r_1 x} \cdot e^{r_2 x} \dots e^{r_n x} \cdot \begin{vmatrix} 1 & \dots & 1 \\ r_1 & & r_n \\ \vdots & & \vdots \\ r_1^{n-1} & \dots & r_n^{n-1} \end{vmatrix}$$

$$= \underbrace{e^{x(r_1 + \dots + r_n)}}_{>0} \cdot \underbrace{\prod_{1 \le i < j \le n} (r_i - r_j)}_{\neq 0} \neq 0.$$

$\Rightarrow \{y_1, \dots, y_n\}$ is a fundam. syst. of sol.

$$\boxed{y_0(x) = c_1 y_1 + \dots + c_n y_n = c_1 e^{r_1 x} + \dots + c_n e^{r_n x}, \quad c_1, \dots, c_n \in \mathbb{R}}$$

2. Case of multiple roots.

Proposition.

If $r \in \mathbb{R}$ is a sol. of (6) with multiplicity $\mu > 1$ then:

$$y_1(x) = e^{rx}$$

$$y_2(x) = x \cdot e^{rx}$$

$$\vdots$$

$$y_\mu(x) = x^{\mu-1} e^{rx}$$

are solutions of (5).

Proof. $y_2(x) = x \cdot e^{rx}$ is a sol. of (5)

r is a sol. of (6) with multiplicity μ $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} P(r) = 0 \\ P'(r) = 0 \\ \vdots \\ P^{(\mu-1)}(r) = 0 \\ P^{(\mu)}(r) \neq 0 \end{cases}$$

$$y_2'(x) = e^{rx} + x.r\,e^{rx}$$

$$y_2''(x) = r.e^{rx} + r e^{rx} + x.r^2 e^{rx} = 2.r e^{rx} + x.r^2 e^{rx}$$

$$y_2'''(x) = \underline{2r^2 e^{rx}} + \underline{r^2 e^{rx}} + x.r^3 e^{rx} = 3r^2 e^{rx} + x.r^3 e^{rx}$$

$$\vdots$$

$$\boxed{y_2^{(k)}(x) = k.r^{k-1}.e^{rx} + x.r^k.e^{rx} \ , \ k = \overline{0,n}}$$

$$y_2^{(n)} + a_1.y_2^{(n-1)} + \ldots + a_n y_2 \overset{?}{=} 0$$

$$n.r^{n-1}.e^{rx} + x.r^n.e^{rx} +$$

$$+ a_1\left((n-1).r^{n-2}.e^{rx} + x.r^{n-1}.e^{rx}\right) + \ldots +$$

$$+ a_{n-1}\left(e^{rx} + x.r\,e^{rx}\right) +$$

$$+ a_n x\,e^{rx} =$$

$$= e^{rx}\Big[\underbrace{n.r^{n-1} + a_1.(n-1)r^{n-2} + \ldots + a_{n-1}}_{P'(r)=0} + x\,(\underbrace{r^n + a_1 r^{n-1} + \ldots + a_n}_{P(r)=0})\Big] =$$

$$= 0$$

3. Case of complex roots.

Proposition

If. $r=\alpha+i\beta\in\mathbb{C}$ is a complex solution of (6) then:

$\Rightarrow y_1(x)=e^{\alpha x}\cdot\cos\beta x$ and $y_2(x)=e^{\alpha x}\cdot\sin\beta x$

are solutions of (5).

Proof. $r=\alpha+i\beta$ is a sol. of (6) then $\bar{r}=\alpha-i\beta$ is a sol. of (6)

Lemma. $y=\underline{u(x)+iv(x)}$, $y: I\to\mathbb{C}$, is a sol. of eq (5) $\Leftrightarrow$ $u(x), v(x)$ are sol. of (5).

$r=\alpha+i\beta$ is a sol. of (6) $\Rightarrow$ $y\underline{(x)=e^{rx}}$ is a solution of (5).

$$y(x)=e^{rx}=e^{(\alpha+i\beta)x}=e^{\alpha x+i\beta x}=e^{\alpha x}\cdot e^{i\beta x}=$$

$$\boxed{e^{i\cdot t}=\cos t+i\sin t}$$

$$=e^{\alpha x}(\cos\beta x+i\cdot\sin\beta x).$$

$\Rightarrow$ $\left.\begin{array}{l} \operatorname{Re} y(x) = e^{\alpha x} \cdot \cos\beta x \\ \operatorname{Im} y(x) = e^{\alpha x} \cdot \sin\beta x \end{array}\right.$ are sol. of (5).

<u>Proposition</u>. If $r = \alpha + i\beta \in \mathbb{C}$ is a complex root of (6) with multiplicity $\mu > 1$ then:

$$y_1(x) = e^{\alpha x}\cos\beta x \qquad y_2(x) = e^{\alpha x}\sin\beta x$$

$$y_3(x) = x\, e^{\alpha x}\cos\beta x \qquad y_4(x) = x \cdot e^{\alpha x}\sin\beta x$$

$$\vdots$$

$$y_{2\mu-1}(x) = x^{\mu-1} \cdot e^{\alpha x}\cos\beta x \,, \quad y_{2\mu}(x) = x^{\mu-1} e^{\alpha x}\sin\beta x$$

are solutions of (5).

The algorithm of solving linear homog. diff. eq. with const. coeff

1. write the characteristic eq.

$$r^n + a_1 r^{n-1} + \dots + a_{n-1} \cdot r + a_n = 0$$

2. solve the charact. eq.

- if $r \in \mathbb{R}$ is a root with multiplicity μ then:

$$y_1(x) = e^{rx}$$
$$\vdots$$
$$y_\mu(x) = x^{\mu-1} \cdot e^{rx}$$

- if $r = \alpha + i\beta \in \mathbb{C}$ with multiplicity μ then:

$$y_1(x) = e^{\alpha x} \cos \beta x \qquad y_2(x) = e^{\alpha x} \sin \beta x$$
$$\vdots$$
$$y_{2\mu-1}(x) = x^{\mu-1} e^{\alpha x} \cos \beta x, \quad y_{2\mu}(x) = x^{\mu-1} e^{\alpha x} \sin \beta x$$

$\Rightarrow y_1(x), \dots, y_n(x)$ fundam. system. of sol.

3. write the gen. sol. of the homog. eq.

$$\boxed{y_0 = c_1 y_1 + \dots + c_n y_n \quad , c_1, \dots, c_n \in \mathbb{R}}$$

The nonhomogeneous case

$y = y_0 + y_p$

y_0 – the gen. sol. of. eq (5)

y_p – is a partic. sol. of eq. (4) which can be found using the variation of the constants method.