Lecture 9

Systems of linear diff. equations

$$\begin{cases} y_1' = a_{11}(x)\cdot y_1 + a_{12}(x)\cdot y_2 + \dots + a_{1n}(x)\cdot y_n + b_1(x) \\ y_2' = a_{21}(x)\cdot y_1 + \dots + a_{2n}(x)\cdot y_n + b_2(x) \\ \vdots \\ y_n' = a_{n1}(x)\cdot y_1 + \dots + a_{nn}(x)\cdot y_n + b_n(x) \end{cases}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\Rightarrow \boxed{Y' = A\cdot Y + B}$ the vectorial form of the system.

$A \in C(I, \mathcal{M}_{nn}(\mathbb{R}))$

$B \in C(I, \mathbb{R}^n)$

(1) $\underline{Y}' - A \cdot \underline{Y} = B$ the nonhomogeneous system.

(2) $\underline{Y}' - A \cdot \underline{Y} = 0$ the homogeneous system

Theorem 1. The IVP:

$$\begin{cases} \underline{Y}' - A \cdot \underline{Y} = B \\ \underline{Y}(a) = r \end{cases}, \quad r \in \mathbb{R}^n$$

has an unique solution.

The homogeneous case

(2) $\underline{Y}' - A\underline{Y} = 0$

$$S_0 = \{ \underline{Y} \in C^1(I, \mathbb{R}^n) \mid \underline{Y}' - A\underline{Y} = 0 \}$$

$$L : C^1(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$$

$$\underline{Y} \mapsto L(\underline{Y})$$

$$L(\underline{Y}) = \underline{Y}' - A \cdot \underline{Y}.$$

the operator L is a linear operator.

L is linear $\Leftrightarrow$ $Y^1, Y^2 \in C^1(I, \mathbb{R}^n)$, $\alpha, \beta \in \mathbb{R}$ then

$$L(\alpha \cdot Y^1 + \beta \cdot Y^2) = \alpha \cdot L(Y^1) + \beta L(Y^2)$$

(2) $\Leftrightarrow$ $L(Y) = 0$

$$\boxed{S_0 = \ker L}$$

<u>Theorem</u> 2. S_0 is a linear subspace of the linear space $C^1(I, \mathbb{R}^n)$ with $\dim S_0 = n$.

<u>Proof</u>. $\left.\begin{array}{l} S_0 = \ker L \\ L \text{ is a linear operator} \end{array}\right\} \Rightarrow S_0$ is a linear subspace of $C^1(I, \mathbb{R}^n)$.

Let us denote by $Y(\cdot, a, r)$ the solution of the IVP:

$$\begin{cases} L(Y) = 0 \\ Y(a) = r \end{cases} \qquad I = [a, b]$$

$$\varphi : \mathbb{R}^n \to S_0$$

$$r \mapsto Y(\cdot; a, r)$$

from Th. 1 $\Rightarrow$ φ is bijective

φ is a linear isomorphis of linear spaces.

$$\Longleftrightarrow \quad \varphi(\lambda_1 r^1 + \lambda_2 r^2) \overset{?}{=} \lambda_1 \varphi(r^1) + \lambda_2 \varphi(r^2) \quad \forall \lambda_1, \lambda_2 \in \mathbb{R}, \; r^1, r^2 \in \mathbb{R}^n$$

$\varphi(\lambda_1 r^1 + \lambda_2 r^2)$ is the sol. $\mathcal{Y}(\cdot; a, \lambda_1 r^1 + \lambda_2 r^2)$

$$\begin{cases} L(\mathcal{Y}) = 0 \\ \mathcal{Y}(a) = \lambda_1 r^1 + \lambda_2 r^2 \end{cases} \quad (3)$$

$\varphi(r^1)$ is the sol $\mathcal{Y}(\cdot; a, r^1)$

$$\begin{cases} L(\mathcal{Y}) = 0 \\ \mathcal{Y}(a) = r^1 \end{cases}$$

$\varphi(r^2)$ is the sol $\mathcal{Y}(\cdot; a, r^2)$

$$\begin{cases} L(\mathcal{Y}) = 0 \\ \mathcal{Y}(a) = r^2 \end{cases}$$

$$\left. \begin{array}{l} V = \lambda_1 \cdot \mathcal{Y}(\cdot; a, r^1) + \lambda_2 \cdot \mathcal{Y}(\cdot; a, r^2) \in S_0 \\ V(a) = \lambda_1 r^1 + \lambda_2 r^2 \end{array} \right\} \Rightarrow V \text{ is a solution of the IVP } (3)$$

$$\overset{\text{Th. 1}}{\Longrightarrow} \quad V = \mathcal{Y}(\cdot, a, \lambda_1 r^1 + \lambda_2 r^2) \Rightarrow \varphi(\lambda_1 r^1 + \lambda_2 r^2) = \lambda_1 \cdot \varphi(r^1) + \lambda_2 \varphi(r^2)$$

$\Rightarrow \boxed{\dim S_0 = \dim \ker L = n} \Rightarrow \exists \{\underline{Y}^1, \underline{Y}^2, \dots, \underline{Y}^n\} \subset S_0$

a basis in S_0

$\forall \underline{Y} \in S_0 \quad \exists\, c_1, \dots, c_n \in \mathbb{R}$ such that:

$$\underline{Y} = c_1 \underline{Y}^1 + c_2 \underline{Y}^2 + \dots + c_n \cdot \underline{Y}^n$$

we denote by $U = (\underline{Y}^1\ \underline{Y}^2 \dots \underline{Y}^n) \Rightarrow$

$$\Rightarrow \boxed{\underline{Y} = U \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}$$

$\{\underline{Y}^1, \underline{Y}^2, \dots, \underline{Y}^n\}$ is a basis in $S_0 \Leftrightarrow \{\underline{Y}^1, \underline{Y}^2, \dots, \underline{Y}^n\}$ is called a fundamental system of solutions.

the matrix $U = (\underline{Y}^1\ \underline{Y}^2 \dots \underline{Y}^n)$ is called the fundamental matrix of solutions.

To solve the system (2) means to find a fundamental matrix of solutions.

$\{\underline{Y}^1, \dots, \underline{Y}^n\} \subset S_0$ is a basis in $S_0 \iff \{\underline{Y}^1, \dots, \underline{Y}^n\}$ is linearly independent system of functions.

$\underline{Y}^1, \dots, \underline{Y}^n$ are linearly dependent $\iff$ if $\exists\ (c_1, \dots, c_n) \neq 0$ such that $c_1 \underline{Y}^1 + \dots + c_n \underline{Y}^n = 0$

$\underline{Y}^1, \dots, \underline{Y}^n$ are linearly independent $\iff$
$\iff \quad c_1 \underline{Y}^1 + \dots + c_n \underline{Y}^n = 0 \implies c_1 = c_2 = \dots = c_n = 0$

$$W(x; \underline{Y}^1, \dots, \underline{Y}^n) = \begin{vmatrix} y_1^1 & y_1^2 & \cdots & y_1^n \\ y_2^1 & y_2^2 & & y_2^n \\ \vdots & \vdots & & \vdots \\ y_n^1 & y_n^2 & & y_n^n \end{vmatrix}$$ the wronskian of $\underline{Y}^1, \dots, \underline{Y}^n$

$$\underline{Y}^1 = \begin{pmatrix} y_1^1 \\ \vdots \\ y_n^1 \end{pmatrix}, \dots, \underline{Y}^n = \begin{pmatrix} y_1^n \\ \vdots \\ y_n^n \end{pmatrix}$$

Theorem 3:

a) if $Y^1, \dots, Y^n \in C([a,b], \mathbb{R}^n)$ are linearly dependent $\Rightarrow$

$\Rightarrow W(\cdot\,; Y^1, Y^2, \dots, Y^n) \equiv 0$

b) If $Y^1, \dots, Y^n \in S_0 = \ker L$ are linearly independent $\Rightarrow$

$\Rightarrow W(x; Y^1, \dots, Y^n) \neq 0$, $\forall x \in [a,b]$.

Proof. a) $Y^1, \dots, Y^n$ are linearly dependent $\Rightarrow$ one function is a linear combination of other $n-1$ functions

$\Rightarrow$ one column of W is a linear combination of other $n-1$ columns of W

$\Rightarrow W(x; Y^1, \dots, Y^n) = 0$, $\forall x \in [a,b]$.

b) suppose $\exists\, x_0 \in [a,b]$ such that $W(x_0; Y^1, \dots, Y^n) = 0$.

we consider the system:

$$\underbrace{\begin{pmatrix} Y^1(x_0) & Y^2(x_0) & \dots & Y^n(x_0) \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0. \quad (4)$$

$\det A = W(x_0; Y^1, \dots Y^n) = 0 \Rightarrow$

$\Rightarrow \exists \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ solution of the system (4).

we construct the function

$$\tilde{Y} = \begin{pmatrix} Y^1 & \dots & Y^n \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix} \Rightarrow \tilde{Y} \in S_0 \Rightarrow L(\tilde{Y}) = 0$$

$$\tilde{Y}(x_0) = \begin{pmatrix} Y^1(x_0) & \dots & Y^n(x_0) \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_n \end{pmatrix} = 0$$

$\tilde{Y}$ is a solution of the IVP

$\begin{cases} L(Y) = 0 \\ Y(x_0) = 0 \end{cases}$ but $Y \equiv 0$ is also a sol of this IVP

$\overset{\text{Th.1.}}{\Rightarrow} \tilde{Y} \equiv 0 \Rightarrow \left. \begin{array}{l} \tilde{c}_1 Y^1 + \dots + \tilde{c}_n Y^n = 0 \\ \text{with } \begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \right\} \Rightarrow Y^1, \dots, Y^n$ are linearly dependent

$\Rightarrow$ contradiction

In S_0 we have the following possibilities:

- if $Y^1, \dots, Y^n \in S_0$ are linearly dependent $\Rightarrow$

$$\Rightarrow W(x; Y^1, \dots, Y^n) = 0, \ \forall x \in [a,b]$$

- if $Y^1, \dots, Y^n \in S_0$ are linearly independent $\Rightarrow$

$$\Rightarrow W(x; Y^1, \dots, Y^n) \neq 0, \ \forall x \in [a,b].$$

Theorem 4. (The wronskian criterion)

$\{Y^1, \dots, Y^n\}$ is a fundamental system of solutions for (2) $\Leftrightarrow \exists x_0 \in [a,b]$ such that $W(x_0; Y^1, \dots, Y^n) \neq 0$.

The nonhomogeneous case

(1) $Y' - AY = B$. $\quad A \in C([a,b], \mathcal{M}_n(\mathbb{R}))$
$B \in C([a,b], \mathbb{R}^n)$

$(1) \Leftrightarrow L(Y) = B$.

$S = \{ Y \in C^1([a,b], \mathbb{R}^n) \mid L(Y) = B \}$ the solution set.

$S = S_0 + \{Y^p\} = \ker L + \{Y^p\}$

where Y^p is a particular solution of (1).

if. U is a fundamental matrix of (2). then the general solution of (1)

$$\boxed{Y = U \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + Y^p, \quad c_1, \dots, c_n \in \mathbb{R}}$$

$\boxed{Y = Y^0 + Y^p}$ where Y^0 is the gen. sol. of (2)
Y^p is a partic. sol. of (1).

The variation of the constants method

$Y^P = ?$

We try to find $Y^P(x) = U(x) \cdot \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}$

$(Y^P)' - A \cdot Y^P = B.$

$\left(1. \left(U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right)' = U' \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} + U \cdot \begin{pmatrix} \varphi_1' \\ \vdots \\ \varphi_n' \end{pmatrix} \right)$

$\left(2. \; U' - A \cdot U = 0 \right)$

$$\Rightarrow \left(U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right)' - A \cdot \left(U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right) = B.$$

$$U' \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} + U \cdot \begin{pmatrix} \varphi_1' \\ \vdots \\ \varphi_n' \end{pmatrix} - A \cdot U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} = B$$

$$\underbrace{(U' - A \cdot U)}_{=0} \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} + U \cdot \begin{pmatrix} \varphi_1' \\ \vdots \\ \varphi_n' \end{pmatrix} = B \quad \Rightarrow$$

$$U^{-1} \cdot \Big| \; U \cdot \begin{pmatrix} \varphi_1' \\ \vdots \\ \varphi_n' \end{pmatrix} = B. \quad \Longrightarrow$$

$\det U(x) = W(x; \underline{y}^1, \dots, \underline{y}^n) \neq 0.$

U is a fundam. matrix of solutions

$$\Longrightarrow \begin{pmatrix} \varphi_1' \\ \vdots \\ \varphi_n' \end{pmatrix} = U^{-1} \cdot B \qquad \begin{pmatrix} \varphi_1'(x) \\ \vdots \\ \varphi_n'(x) \end{pmatrix} = U^{-1}(x) \cdot B(x)$$

$$\Longrightarrow \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} = \int_{x_0}^{x} U^{-1}(s) \cdot B(s)\, ds$$

$$\underline{y}^P = U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \Longrightarrow \boxed{\underline{y}^P(x) = U(x) \cdot \int_{x_0}^{x} U^{-1}(s) \cdot B(s)\, ds}$$

the general sol. of (1):

$\underline{y} = \underline{y}^0 + \underline{y}^P \quad \Longrightarrow$

$$\boxed{Y(x) = U(x)\cdot\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + U(x)\cdot\int_{x_0}^{x} U^{-1}(s)\cdot B(s)\,ds}$$

$x_0 \in [a,b]$

Remarks

1) $\begin{cases} Y' - AY = B \\ Y(x_0) = V \end{cases}$, $V \in \mathbb{R}^n$

$$Y(x_0) = V \Rightarrow U(x_0)\cdot\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = V \Rightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = U^{-1}(x_0)\cdot V$$

$\Rightarrow$ the solution of IVP is

$$Y(x) = U(x)\cdot U^{-1}(x_0)\cdot V + U(x)\cdot\int_{x_0}^{x} U^{-1}(s)\cdot B(s)\,ds.$$

2) $n = 1$ $\quad y' - p\cdot y = q \qquad p, q \in C([a,b], \mathbb{R})$

$$U(x) = e^{\int_{x_0}^{x} p(s)\,ds}$$

$$U^{-1}(x) = e^{-\int_{x_0}^{x} p(s)\,ds}$$

$$y(x) = c \cdot e^{\int_{x_0}^{x} p(s)\,ds} + e^{\int_{x_0}^{x} p(s)\,ds} \cdot \left(\int_{x_0}^{x} \left(e^{-\int_{x_0}^{s} p(t)\,dt} \cdot g(s) \right) ds \right)$$