Lecture 10

Linear systems with constant coefficients

$$(1)\quad \begin{cases} y_1' = a_{11}\cdot y_1 + \dots + a_{1n}\cdot y_n + b_1 \\ \vdots \\ y_n' = a_{n1}\cdot y_1 + \dots + a_{nn} y_n + b_n \end{cases}$$

$$\underline{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad A = (a_{ij})_{1\le i,j\le n} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$A \in \mathcal{M}_n(\mathbb{R}), \quad B \in C(I, \mathbb{R}^n)$$

$$\boxed{\underline{Y}' = A\cdot \underline{Y} + B}$$

The homogeneous case

(2) $Y' = AY.$

$y' = ay$

$y(x) = c\cdot e^{ax}, c \in \mathbb{R}$, gen. sol.

$\underline{\underline{y(x) = e^{ax}}}$ is the fundam. sol.

$$\boxed{(e^{ax})' = a\cdot e^{ax}}$$

I The Exponential Matrix Method

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$A \in \mathcal{M}_n(\mathbb{R})$

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} + \dots$$

$$\mathbb{R} \longmapsto \mathcal{M}_n(\mathbb{R})$$
$$x \longmapsto e^{xA}$$ the exponential matrix function

$$\underline{\underline{e^{xA}}} = I + \frac{x\cdot A}{1!} + \frac{x^2 A^2}{2!} + \dots + \frac{x^n\cdot A^n}{n!} + \dots$$

$e^{0\cdot A} = I$, $\boxed{(e^{xA})' = A\cdot e^{xA}}$

$U(x) = e^{xA}$ $\qquad U(0) = I$ $\qquad \Rightarrow U(x) = e^{xA}$ is a fundamental matrix of solution

$\det U(0) = 1 \neq 0$

$\Rightarrow$ the gen. solution of (2)

$$Y = U(x) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\boxed{Y(x) = e^{xA} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad c_1, \dots, c_n \in \mathbb{R}}$$

II The reduction method to a n order linear diff. eq with constant coefficients

$$(2) \quad \begin{cases} y_1' = a_{11} y_1 + \dots + a_{1n} \cdot y_n \\ \vdots \\ y_n' = a_{n1} y_1 + \dots + a_{nn} y_n \end{cases}$$

$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ a sol. of (2) $\Rightarrow Y \in C^{\infty}$

we choose one equation of the system (2).

$$y_1' = a_{11} \cdot y_1 + \dots + a_{1n} \cdot y_n$$

we derivate this eq. with respect to x.

$$y_1'' = a_{11} y_1' + \dots + a_{1n} \cdot y_n' = a_{11} (a_{11} y_1 + \dots + a_{1n} y_n) + \dots +$$

$$+ a_{1n} (a_{n1} y_1 + \dots + a_{nn} y_n)$$

$$\Rightarrow \dots \Rightarrow$$

$$\boxed{y_1'' = a_{11}^1 y_1 + \dots + a_{1n}^1 y_n}$$

we derivate again with respect to x.

$$y_1''' = a_{11}^1 \underline{\underline{y_1'}} + \dots + a_{1n}^1 \cdot \underline{\underline{y_n'}} = \dots =$$

we replace
with relations from
the system (2)

$$\boxed{y_1''' = a_{11}^2 y_1 + \dots + a_{1n}^2 y_n}$$

we continue with this procedure until we get $y_1^{(n)}$

$$y_1^{(i)} = a_{11}^{i-1} y_1 + \dots + a_{1n}^{i-1} \cdot y_n \,, \quad i = \overline{2,n}$$

$$\begin{cases} a_{12} \cdot y_2 + \dots + a_{1n} y_n = y_1' - a_{11} y_1 \\ a_{12}^1 y_2 + \dots + a_{1n}^1 y_n = y_1'' - a_{11}^1 \cdot y_1 \\ \vdots \\ a_{12}^{n-2} y_2 + \dots + a_{1n}^{n-2} y_n = y_1^{(n-1)} - a_{11}^{n-2} y_1 \end{cases}$$

$n-1$ system of eq.

$\Rightarrow$ we solve this system with respect to $y_2, y_3, \dots, y_n$

$$\Rightarrow \dots \Rightarrow \boxed{y_k = \alpha_{k1} y_1 + \alpha_{k2} \cdot y_1' + \dots + \alpha_{k\,n-1} y_1^{(n-1)} \,, \quad k = \overline{2,n}} \quad (3)$$

these relations are replaced in the last relation

$$y_1^{(n)} = a_{11}^{n-1} \cdot y_1 + \dots + a_{1n}^{n-1} \cdot y_n$$

$$\Rightarrow \dots \Rightarrow \boxed{y_1^{(n)} + b_1 \cdot y_1^{(n-1)} + \dots + b_n y_1 = 0}$$

a homogeneous linear diff. eq. with const. coeff.

$r^n + b_1 \cdot r^{n-1} + \dots + b_n = 0$ the characteristic eq.

$\Rightarrow$ $\varphi_1, \dots, \varphi_n$ the fundam. system solutions

$$\Rightarrow \boxed{y_1(x) = c_1 \varphi_1(x) + \dots + c_n \varphi_n(x)}$$

replaceing $y_1(x)$ in (3) $\to$ $y_2(x), \dots, y_n(x)$.

III The characteristic equation method

$$\underline{Y}' = A \cdot \underline{Y}$$

$$\underline{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \cdot e^{\lambda x} \\ \alpha_2 e^{\lambda x} \\ \vdots \\ \alpha_n \cdot e^{\lambda x} \end{pmatrix}, \quad \underline{Y} \neq 0 \iff \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\underline{Y}' - A \cdot \underline{Y} = 0 \Rightarrow \begin{pmatrix} \alpha_1 \cdot \lambda \cdot e^{\lambda x} \\ \vdots \\ \alpha_n \cdot \lambda \cdot e^{\lambda x} \end{pmatrix} - A \cdot \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \vdots \\ \alpha_n e^{\lambda x} \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha_1 \cdot \lambda \\ \vdots \\ \alpha_n \cdot \lambda \end{pmatrix} \cdot e^{\lambda x} - A \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \cdot e^{\lambda x} = 0 \quad / : e^{\lambda x}$$

$$\begin{pmatrix} \alpha_1 \cdot \lambda \\ \vdots \\ \alpha_n \cdot \lambda \end{pmatrix} - A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$$\lambda \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} - A \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$$(4) \quad \left(\lambda \cdot I - A \right) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \neq 0 \Rightarrow (5) \quad \boxed{\det(\lambda I - A) = 0}$$ the characteristic equation

the solutions of the characteristic eq. (5) are the eigenvalues of A.

the charact. eq. (5) is a n-degree polynomial eq.

a) The case of simple real eigenvalues

$\lambda_1, \dots, \lambda_n$ eigenvalues of A. $\lambda_i \neq \lambda_j$, $i \neq j$.

for each λ_j, $j = \overline{1,n}$, we construct a non-zero solution of the system (4)

$$\begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_n^j \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad j = \overline{1,n}.$$

$$\Rightarrow \underline{Y}^j = \begin{pmatrix} \alpha_1^j e^{\lambda_j x} \\ \vdots \\ \alpha_n^j \cdot e^{\lambda_j x} \end{pmatrix}, \quad j = \overline{1,n}$$

$\Rightarrow U(x) = \left(\underline{Y}^1 \ \underline{Y}^2 \dots \underline{Y}^n \right)$ a fundam. matrix of sol.

b) The case of complex eigenvalue

$\lambda = \alpha + i\beta$ is an eigenvalue of A.

$\Rightarrow \bar{\lambda} = \alpha - i\beta$ is an eigenvalue of A.

$z(x) = z_1(x) + i \cdot z_2(x)$

$z(x)$ is a sol. of the system (2) $\Leftrightarrow$ $z_1(x), z_2(x)$ are sol. of (2)

$$z(x) = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \vdots \\ \alpha_n e^{\lambda x} \end{pmatrix} \quad \alpha_1, \dots, \alpha_n \in \mathbb{C}, \quad \underline{\lambda = \alpha + i\beta.}$$

$\Rightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is a sol. of the system (4) in $\mathbb{C}$

suppose $\alpha_1 = a_1 + ib_1, \dots, \alpha_n = a_n + ib_n$ a nonzero sol. of (4).

$$\Rightarrow z(x) = \begin{pmatrix} (a_1+ib_1) \cdot e^{(\alpha+i\beta)x} \\ \vdots \\ (a_n+ib_n) e^{(\alpha+i\beta)x} \end{pmatrix} = \begin{pmatrix} (a_1+ib_1) e^{\alpha x} \cdot (\cos\beta x + i\sin\beta x) \\ \vdots \\ (a_n+ib_n) e^{\alpha x} (\cos\beta x + i\sin\beta x) \end{pmatrix}$$

$$\boxed{e^{\alpha+ib} = e^{\alpha} \cdot (\cos\beta + i\sin\beta)}$$

$$= \underbrace{\begin{pmatrix} e^{\alpha x}\cdot(a_1\cos\beta x - b_1\sin\beta x) \\ \vdots \\ e^{\alpha x}(a_n\cos\beta x - b_n\sin\beta x) \end{pmatrix}}_{Y^1} + i\cdot \underbrace{\begin{pmatrix} e^{\alpha x}(a_1\sin\beta x + b_1\cos\beta x) \\ \vdots \\ e^{\alpha x}\cdot(a_n\sin\beta x + b_n\cos\beta x) \end{pmatrix}}_{Y^2}$$

$\operatorname{Re} Z(x) = Y^1$, $\operatorname{Im} Z(x) = Y^2$ are sol. of (2).

c) <u>The case of multiple eigenvalues</u>

λ is a multiple real eigenvalue with the multiplicity order $\mu > 1$.

$$\Rightarrow \quad Y^1(x) = e^{\lambda x}\cdot u_1$$

$$Y^2(x) = e^{\lambda x}\left(\frac{x}{1!}u_1 + u_2\right)$$

$$\vdots$$

$$Y^\mu(x) = e^{\lambda x}\left(\frac{x^{\mu-1}}{(\mu-1)!}u_1 + \dots + \frac{x}{1!}u_{\mu-1} + u_\mu\right)$$

where u_1 is a nonzero sol. of $Au_1 = \lambda u_1$

$u_2, \dots, u_\mu$ are nonzero solutions of the systems:

$$\begin{cases} Au_2 = \lambda u_2 + u_1 \\ Au_3 = \lambda u_3 + u_2 \\ \vdots \\ Au_\mu = \lambda u_\mu + u_{\mu-1} \end{cases} \iff$$

$$\iff \begin{cases} (\lambda I - A)u_1 = 0 \\ (\lambda I - A)u_2 = u_1 \\ \vdots \\ (\lambda I - A)u_\mu = u_{\mu-1} \end{cases}$$

$\Rightarrow Y^1, \dots, Y^n$ sol. of (2)

$\Rightarrow U(x) = (Y^1\ Y^2 \dots Y^n)$ the fundam. matrix of sol.

$$\Rightarrow \boxed{Y = U(x) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \ c_1, \dots, c_n \in \mathbb{R}}$$

The nonhomogeneous case

(1) $Y' = A \cdot Y + B$ $\qquad A \in \mathcal{M}_n(\mathbb{R}),\ B \in C(I, \mathbb{R}^n)$

the general solution of (1)

$Y = Y^0 + Y^p$ where Y^0 is the gen. sol. of (2) $Y' = AY$

Y^p is a particular sol. of (1) which can be found using the variation of constants method.

if $U(x)$ is a fundam. matrix of (2)

then $\qquad Y^p(x) = U(x) \cdot \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}$

$$B = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$$