Lecture 11

The dynamical systems generated by scalar autonomous diff. equations

$x = x(t)$ $\quad x' = f(t, x)$ a nonautonomous diff. eq.

(1) $\boxed{x' = f(x)}$ an autonomous diff. eq.

scalar $\rightarrow$ x is one dimensional

Theorem. If $f \in C^1(\mathbb{R})$ then the IVP

$$\begin{cases} x' = f(x) & (1) \\ x(0) = \eta & (2) \end{cases}, \quad \eta \in \mathbb{R}$$

has an unique maximal solution for every $\eta \in \mathbb{R}$.

maximal solution = a solution defined on the largest possible interval.

let's denote by $x(t, \eta)$ the sol. of the IVP (1)+(2)

$x(\cdot, \eta) : I_\eta \rightarrow \mathbb{R}$

$x(\cdot, \eta)$ is maximal $\Leftrightarrow$ I_η is maximal,

$I_\eta = (\alpha_\eta, \beta_\eta)$ – open interval.

$0 \in I_\eta \Rightarrow \alpha_\eta < 0 < \beta_\eta$

$\varphi : W \to \mathbb{R}$

$W = \{ I_\eta \times \mathbb{R} \mid \eta \in \mathbb{R} \}$

$\varphi(t,\eta) = x(t,\eta)$

if $I_\eta = \mathbb{R}, \forall \eta \in \mathbb{R} \Rightarrow$

$\Rightarrow W = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

the function φ is called the flow generated by the eq.(1)

the map $\eta \mapsto \varphi(t,\eta) \to$ the dynamical systems generated by (1).

<u>Properties:</u>

1) $\varphi(0,\eta) = \eta$
2) $\varphi(t+s,\eta) = \varphi(t, \varphi(s,\eta))$, $\forall t,s \in I_\eta$, $\forall \eta \in \mathbb{R}$.
3) φ is continuous

Definition

$$\gamma^+(\eta) = \bigcup_{t\in[0,\beta_\eta)} \varphi(t,\eta) \quad \text{the positive orbit of } \eta$$

$$\gamma^-(\eta) = \bigcup_{t\in(\alpha_\eta,0]} \varphi(t,\eta) \quad \text{the negative orbit of } \eta$$

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = \bigcup_{t\in(\alpha_\eta,\beta_\eta)} \varphi(t,\eta) \quad \text{the orbit of } \eta.$$

The collection of all orbits together with the direction of the flow is called the phase portrait of diff. eq.

Examples

1) $x' = -x \qquad f(x) = -x$

flow: $\begin{cases} x' = -x \\ x(0) = \eta, \ \eta \in \mathbb{R}. \end{cases}$

$$\frac{dx}{dt} = -x \Rightarrow \int \frac{dx}{x} = -\int dt \Rightarrow$$

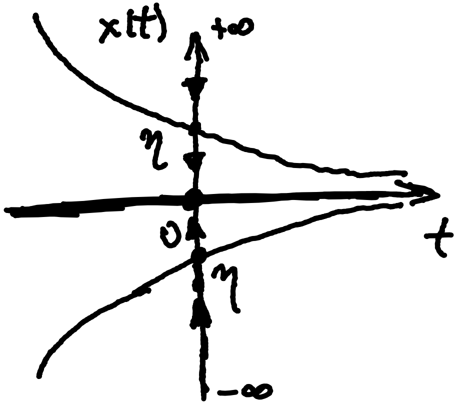
$$\Rightarrow \ln x = -t + \ln c$$

$$\underline{| x(t) = c.e^{-t}, c \in \mathbb{R} |}$$

$$x(0) = \eta \Rightarrow c = \eta \Rightarrow x(t,\eta) = \eta e^{-t}$$

$$I_\eta = \mathbb{R}, \ \forall \eta \in \mathbb{R}$$

$$\left.\begin{array}{l}\varphi(t,\eta) = x(t,\eta) = \eta\cdot e^{-t} \\ \varphi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\end{array}\right\} \longrightarrow \text{the flow generated by the diff. eq.}$$



1. if $\eta = 0 \Rightarrow \varphi(t,0) \equiv 0$

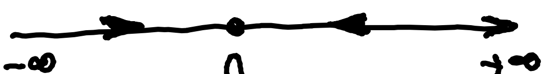
$$\gamma(0) = \bigcup_{t\in\mathbb{R}} \varphi(t,0) = \{0\}$$

2. if $\underline{\eta > 0}$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = (0,\eta]$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = [\eta,+\infty)$$

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = (0,+\infty)$$



phase portrait

3. if $\underline{\eta < 0}$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = [\eta,0)$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = (-\infty,\eta]$$

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = (-\infty,0)$$

2) $x' = x$

flow: $\begin{cases} x' = x \\ x(0) = \eta \end{cases}$

$\frac{dx}{dt} = x \to \int \frac{dx}{x} = \int dt$

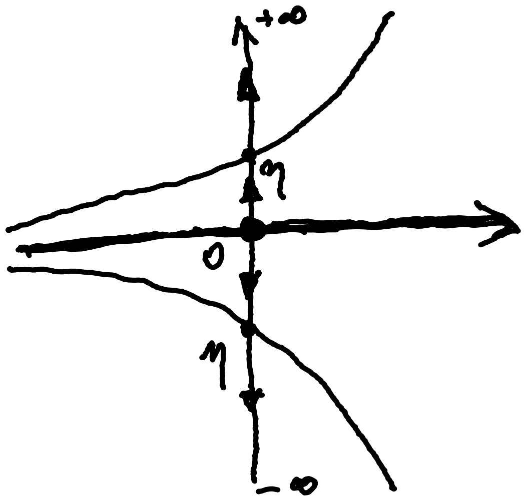
$\ln x = t + \ln c$

$\boxed{x(t) = c \cdot e^t, c \in \mathbb{R}}$

$x(0) = \eta \to c = \eta \to x(t, \eta) = \eta \cdot e^t$

the maximal interval $I_\eta = \mathbb{R}, \forall \eta \in \mathbb{R}$

$\left.\begin{array}{l} \varphi(t, \eta) = x(t, \eta) = \eta \cdot e^t \\ \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \end{array}\right\}$ the flow.



1. $\eta = 0 \to \varphi(t, 0) \equiv 0$

$\gamma(0) = \bigcup_{t \in \mathbb{R}} \varphi(t, 0) = \{0\}$

2. $\eta > 0$

$\gamma^+(\eta) = \bigcup_{t \in [0, +\infty)} \varphi(t, \eta) = [\eta, +\infty)$

$\gamma^-(\eta) = \bigcup_{t \in (-\infty, 0]} \varphi(t, \eta) = (0, \eta]$

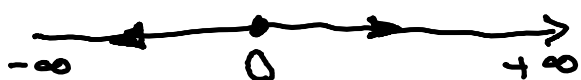
$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = (0, +\infty)$

3. $\eta < 0$

$$\gamma^+(\eta) = \bigcup_{t \in [0,+\infty)} \varphi(t,\eta) = (-\infty, \eta] \quad \longleftarrow$$

$$\gamma^-(\eta) = \bigcup_{t \in (-\infty,0]} \varphi(t,\eta) = [\eta, 0) \quad \longleftarrow$$

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = (-\infty, 0) \quad \longleftarrow$$

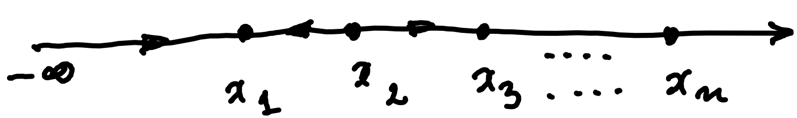


phase portrait.

In general $\quad x' = f(x)$.

$f(x) = 0 \quad \Rightarrow \quad x_1, x_2, \dots, x_n$ real roots.

x	$-\infty$		x_1		x_2		x_3	...	x_n
$f(x)$		+	0	−	0	+	0	....,	0
		$\rightarrow$		$\leftarrow$		$\rightarrow$			



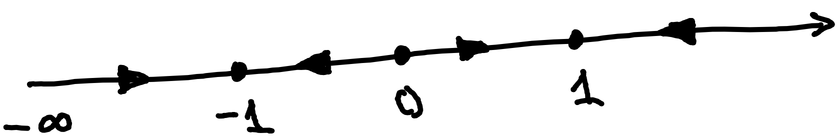
phase portrait

3) $x' = x - x^3 \qquad f(x) = x - x^3$

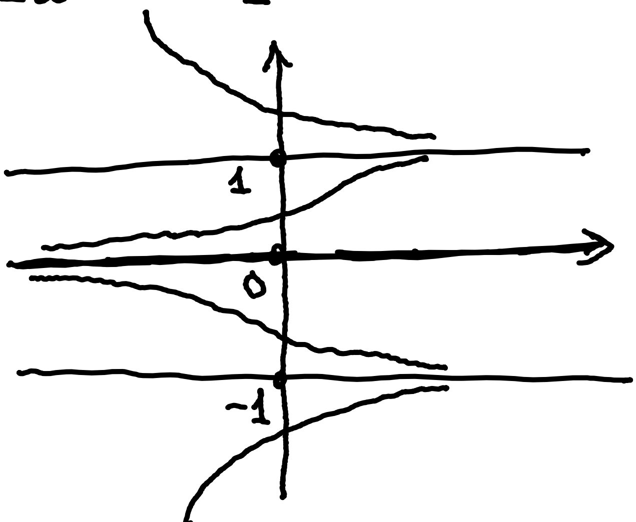
$$f(x) = 0 \Rightarrow x - x^3 = 0 \Rightarrow \underbrace{x}_{0}\underbrace{(1 - x^2)}_{\pm 1} = 0$$

$$x_1 = -1,\ x_2 = 0,\ x_3 = 1$$

x		-1		0		1	
$f(x)$	$+$ $\rightarrow$	0	$-$ $\leftarrow$	0	$+$ $\rightarrow$	0	$-$ $\leftarrow$



phase portrait.



Def. The constant solutions $x(t) \equiv x^*$ of the eq. (1) are called equilibrium solutions (stationary)

The value $x^* \in \mathbb{R}$ is called the equilibrium point.

in the case of our examples:

1) $x' = -x \Rightarrow x^* = 0$ is an eq. point
2) $x' = x \Rightarrow x^* = 0$ is an. eq. point
3) $x' = x - x^3 \Rightarrow x_1^* = -1,\ x_2^* = 0,\ x_3^* = 1$ are eq. points.

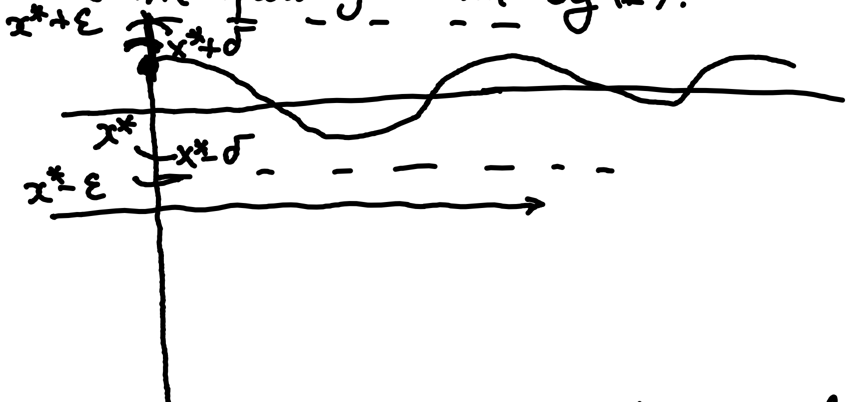
$$\left.\begin{array}{l} x' = f(x) \\ x(t) \equiv x^* \end{array}\right\} \Rightarrow x^* \text{ is a sol. of the eq. } \boxed{f(x) = 0}$$

Def. An equilibrium point $x^* \in \mathbb{R}$ of the eq. (1) is:

a) locally stable $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon)$ such that for every η for which $|\eta - x^*| < \delta$ we have

$$|\varphi(t, \eta) - x^*| < \varepsilon \ , \ \forall t \geq 0$$

where φ is the flow generated by (1).



b) locally asymptotically stable $\Leftrightarrow$ if it is locally stable and $|\varphi(t, \eta) - x^*| \xrightarrow[t \to \infty]{} 0$.

c) unstable $\Leftrightarrow$ if it is not locally stable.

Examples

1) $x' = -x$

$x^* = 0$ is an eq. point.

$\varphi(t,\eta) = \eta e^{-t}$ $\varphi : \mathbb{R}^2 \to \mathbb{R}$

$$|\varphi(t,\eta) - x^*| = |\eta e^{-t}| = |\eta| \cdot \underbrace{e^{-t}}_{\leq 1,\ t \geq 0} \leq |\eta| = |\eta - x^*|$$

for $\varepsilon > 0$ $\exists\, \delta = \varepsilon$ such that if $|\eta - x^*| < \delta = \varepsilon$

$\Rightarrow |\varphi(t,\eta) - x^*| \leq |\eta - x^*| < \delta = \varepsilon$, $\forall t \geq 0 \Rightarrow$

$\Rightarrow x^* = 0$ is locally stable

$$|\varphi(t,\eta) - x^*| = |\eta| \cdot e^{-t} \xrightarrow[t \to +\infty]{} 0$$

$\Rightarrow x^* = 0$ is ~~locally~~ asympt. stable

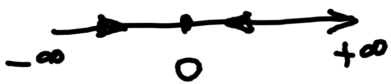
2) $x' = x$ $\quad x^* = 0$ is an. eq. point

$$\varphi(t,\eta) = \eta \cdot e^t \qquad \varphi : \mathbb{R}^2 \to \mathbb{R}.$$

$$|\varphi(t,\eta) - x^*| = |\eta| \cdot e^t \xrightarrow[t \to +\infty]{} +\infty$$

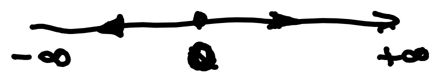
$\Rightarrow x^*$ is unstable.

$x' = -x$



asympt. stable

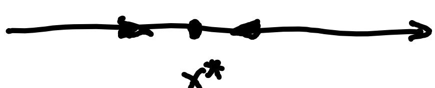
$x' = x$



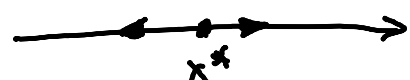
unstable.

$x' = f(x)$

x^* an eq. point

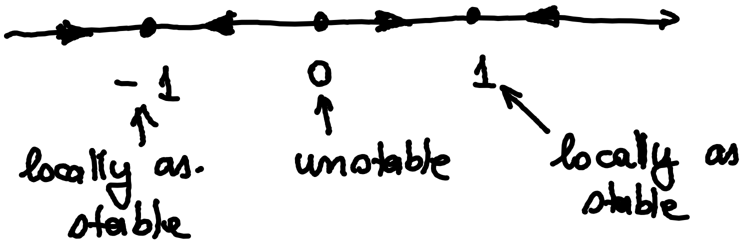


locally as. stable



unstable

3) $x' = x - x^3$ $\qquad x_1^* = -1,\ x_2^* = 0,\ x_3^* = 1$



Theorem. (The Stability in first approximation)
$x^* \in \mathbb{R}$ is an eq. point of (1), $f \in C^1$.

a) if $f'(x^*) < 0 \Rightarrow x^*$ is locally as. stable

b) if $f'(x^*) > 0 \Rightarrow x^*$ is unstable.

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for eq. $x' = x - x^3$

$f(x) = x - x^3$

$f'(x) = 1 - 3x^2$

we have the eq. points $x_1^* = -1,\ x_2^* = 0,\ x_3^* = 1$

$f'(x_1^*) = f'(x_3^*) = f'(\pm 1) = 1 - 3 = -2 < 0 \Rightarrow$

$\Rightarrow x_1^* = -1$ and $x_2^* = 1$ are locally as. stable

$f'(x_2^*) = f'(0) = 1 > 0 \Rightarrow x_2^* = 0$ is unstable.