Lecture 12

Dynamical systems generated by planar autonomous systems

$x(t)$, $y(t)$ unknown functions.

$$(1)\quad \begin{cases} x'(t) = f_1(x,y) \\ y'(t) = f_2(x,y) \end{cases} \qquad f = (f_1, f_2)$$

Theorem. If $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ then the IVP:

$$(2)\quad \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$$

has a unique saturated solution for every $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$.

We denote by $(x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2))$ the unique sol. of IVP (2).

$x(\cdot, \eta_1, \eta_2)$, $y(\cdot, \eta_1, \eta_2) : I_\eta \to \mathbb{R}$

I_η is the maximal interval $\longmapsto I_\eta = (\alpha_\eta, \beta_\eta)$

$0 \in I_\eta \Rightarrow \alpha_\eta < 0 < \beta_\eta$

$W = \{ I_\eta \times \mathbb{R}^2 \mid \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \}$.

<u>The flow</u> generated by (1):

$\varphi : W \to \mathbb{R}^2$

$$\varphi(t, \eta) = \varphi(t, \eta_1, \eta_2) = \big(x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2) \big).$$

The map: $\forall \eta = (\eta_1, \eta_2)$

$\eta \longmapsto \varphi(\cdot, \eta_1, \eta_2)$ is called the dynamical system generated by (1).

Properties of the flow

1. $\varphi(0, \eta_1, \eta_2) = (\eta_1, \eta_2)$, $\forall \eta = (\eta_1, \eta_2) \in \mathbb{R}^2$
2. $\varphi(t+s, \eta_1, \eta_2) = \varphi(t, \varphi(s, \eta_1, \eta_2))$ $\forall t, s \in I_\eta$, $\forall \eta \in \mathbb{R}^2$
3. φ is continuous.

Definition

$\gamma^+(\eta) = \gamma^+(\eta_1, \eta_2) = \bigcup\limits_{t \in [0, \beta_\eta)} \varphi(t, \eta)$ the positive orbit of $\eta = (\eta_1, \eta_2)$

$\gamma^-(\eta) = \gamma^-(\eta_1, \eta_2) = \bigcup\limits_{t \in (\alpha_\eta, 0]} \varphi(t, \eta)$ the negative orbit of $\eta = (\eta_1, \eta_2)$

$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta)$ – the orbit of $\eta = (\eta_1, \eta_2)$.

Phase portrait: is the collection of all orbits together with the developing direction which show the direction in which the flow is changeing when t increases.

Example

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

flow:
$$\begin{cases} x' = y \\ y' = -x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \qquad \eta = (\eta_1, \eta_2) \in \mathbb{R}^2$$

$$x' = y \Rightarrow \left.\begin{matrix} x'' = y' \\ y' = -x \end{matrix}\right\} \Rightarrow x'' = -x$$

$$\boxed{x'' + x = 0} \Rightarrow$$

$\Rightarrow r^2 + 1 = 0$ the char. eq.

$$r_{1,2} = \pm i \qquad \begin{matrix} \alpha = 0 \\ \beta = 1 \end{matrix} \qquad \begin{matrix} x_1(t) = e^{\alpha \cdot t} \cos \beta t = e^{0 \cdot t} \cdot \cos t \\ x_2(t) = e^{\alpha t} \cdot \sin \beta t = \sin t \end{matrix}$$

$$\Rightarrow \boxed{x(t) = c_1 \cos t + c_2 \sin t, \quad c_1, c_2 \in \mathbb{R}}$$

$$y = x' = -c_1 \sin t + c_2 \cos t$$

the gen. sol. of the syst.

$$\begin{cases} x(t) = c_1 \cos t + c_2 \sin t \\ y(t) = -c_1 \sin t + c_2 \cos t \end{cases}, \quad c_1, c_2 \in \mathbb{R}.$$

$x(0)=\eta_1 \Rightarrow c_1=\eta_1$

$y(0)=\eta_2 \Rightarrow c_2=\eta_2$

$$\Rightarrow \begin{cases} x(t,\eta_1,\eta_2) = \eta_1 \cos t + \eta_2 \sin t \\ y(t,\eta_1,\eta_2) = -\eta_1 \sin t + \eta_2 \cos t \end{cases}$$

$x(\cdot,\eta_1,\eta_2),\ y(\cdot,\eta_1,\eta_2) : I_\eta \to \mathbb{R}$

$I_\eta = \mathbb{R}, \ \forall \eta \in \mathbb{R}^2$

$\varphi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi(t,\eta_1,\eta_2) = (x(t,\eta_1,\eta_2), y(t,\eta_1,\eta_2)) =$$
$$= (\eta_1 \cos t + \eta_2 \sin t, -\eta_1 \sin t + \eta_2 \cos t)$$

φ – the flow generated by the system.

Orbits:

1. $(\eta_1, \eta_2) = (0,0) \Rightarrow \varphi(0,0) = (0,0)$.

$$\gamma(0,0) = \bigcup_{t\in\mathbb{R}} \varphi(0,0) = \bigcup_{t\in\mathbb{R}} (0,0) = \{(0,0)\}.$$

2. $(\eta_1, \eta_2) \neq (0,0)$

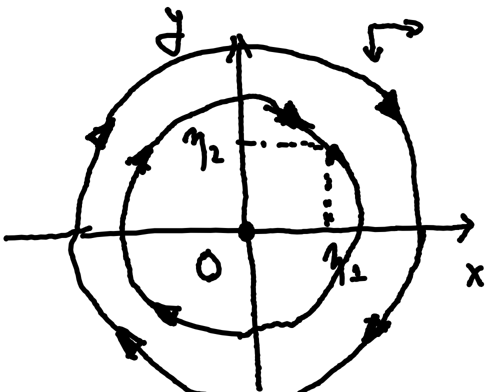
$$\gamma(\eta_1,\eta_2) = \bigcup_{t\in\mathbb{R}} \varphi(t,\eta_1,\eta_2) =$$

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

$$= \bigcup_{t\in\mathbb{R}} (\underbrace{\eta_1 \cos t + \eta_2 \sin t}_{x}, \underbrace{-\eta_1 \sin t + \eta_2 \cos t}_{y}).$$

$$\begin{cases} x = \eta_1 \cos t + \eta_2 \sin t \\ y = -\eta_1 \sin t + \eta_2 \cos t \end{cases}, \ t\in\mathbb{R}.$$

a curve given by the parametric eq.



phase portrait.

$$x^2 + y^2 = \eta_1^2 \cos^2 t + \cancel{2\eta_1\eta_2 \cos t \sin t} + \eta_2^2 \sin^2 t$$
$$\eta_1^2 \sin^2 t - \cancel{2\eta_1\eta_2 \sin t \cos t} + \eta_2^2 \cos^2 t$$

$x^2 + y^2 = \eta_1^2 + \eta_2^2$ the orbit is a circle centered in $(0,0)$ with radius $\sqrt{\eta_1^2 + \eta_2^2}$.

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = f_1(x,y) \\ \frac{dy}{dt} = f_2(x,y) \end{cases} \Rightarrow \boxed{\underbrace{\frac{dx}{dy}}_{x'(y)} = \frac{f_1(x,y)}{f_2(x,y)}}$$

the diff. eq. of the orbits.

or: $\underbrace{\frac{dy}{dx}}_{y'(x)} = \frac{f_2(x,y)}{f_1(x,y)}$

$$\begin{cases} x' = y \\ y' = -x \end{cases} \Rightarrow \begin{array}{l} f_1(x,y) = y \\ f_2(x,y) = -x \end{array}$$

$$\frac{dx}{dy} = \frac{y}{-x} \Rightarrow y\,dy = -x\,dx \mid \cdot 2.$$

$$\int 2y\,dy = \int -2x\,dx$$

$$y^2 = -x^2 + c \Rightarrow \boxed{x^2 + y^2 = c},\ c \in \mathbb{R}.$$

the implicit eq. of the orbits.

Definition

A constant sol. of the syst. (1) is called an equilibrium sol.

$$\begin{cases} x(t) \equiv x^* \\ y(t) \equiv y^* \end{cases}$$

$(x^*, y^*) \in \mathbb{R}^2$ – equilibrium point.

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \rightarrow (x^*, y^*) \text{ is a real solution of the syst.}$$

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases} \qquad X^* = (x^*, y^*)$$

Definition An equilibrium point $X^*(x^*, y^*)$ of the syst (1) is called:

a) locally stable if $\forall \varepsilon > 0 \; \exists \delta(\varepsilon) > 0$ such that if $\|\eta - X^*\|_{\mathbb{R}^2} \leq \delta$ then $\|\varphi(t, \eta) - X^*\|_{\mathbb{R}^2} \leq \varepsilon, \; \forall t > 0$

b) locally asymptotically stable if it is stable and

$$\|\varphi(t, \eta) - X^*\| \xrightarrow[t \to +\infty]{} 0$$

c) unstable if it is not stable.

Linear case

$$(2)\quad \begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}).$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$X^*(0,0)$ is an equilibrium point of (2).

the characteristic equation

$$(3)\quad \boxed{\det(\lambda I_2 - A) = 0}$$

Theorem (The Stability Theorem

Let's consider the syst. (2).

a) If $Re\lambda < 0$, $\forall \lambda$ eigenvalue of $A \Rightarrow X^*(0,0)$ is asympt. stable

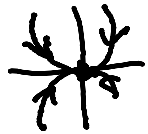
b) If $Re\lambda \leq 0$, $\forall \lambda$ eigenvalue of A, but $Re\lambda = 0$ holds for simple eigenvalue $\Rightarrow X^*(0,0)$ is locally stable

c) If $\exists \lambda$ with $Re\lambda > 0$ or $Re\lambda = 0$ and λ is not simple $\Rightarrow X^*(0,0)$ is unstable.

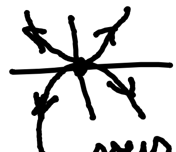
The classification of $X^*(0,0)$:

I $\lambda_1, \lambda_2 \in \mathbb{R}$.

- if $\lambda_1 \leq \lambda_2 < 0 \Rightarrow X^*(0,0)$ is asymt. stable node type (sink node)
- if $\lambda_1 \geq \lambda_2 > 0 \rightarrow (0,0)$ is unstable node type (source node).

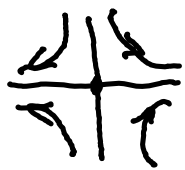


sink node



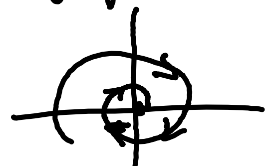
source node

- if $\lambda_1 < 0 < \lambda_2 \Rightarrow (0,0)$ is saddle point type. (always unstable)

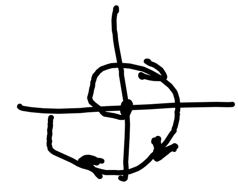


II $\lambda_{1,2} = \alpha \pm i\beta \in \mathbb{C}$

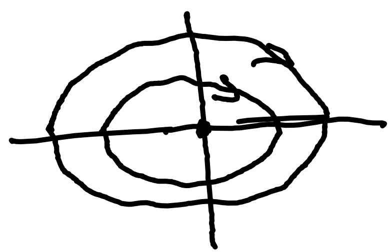
- if $\alpha \neq 0 \Rightarrow (0,0)$ is focus type (spiral type)
  - if $\alpha < 0 \Rightarrow$ asympt. stable focus (stable spiral)



  - if $\alpha > 0 \rightarrow$ unstable focus



- if $\alpha = 0 \rightarrow (0,0)$ is center type (always locally stable)



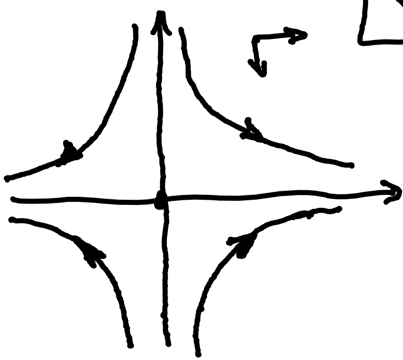
Examples

1) $\begin{cases} x' = x \\ y' = -2y \end{cases}$

$$\frac{dx}{dy} = \frac{x}{-2y} \Rightarrow \int \frac{dy}{y} = \int \frac{-2dx}{x}$$

$$\ln y = -2 \ln x + \ln c .$$

$$\boxed{y = \frac{c}{x^2} ,\ c \in \mathbb{R}}$$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

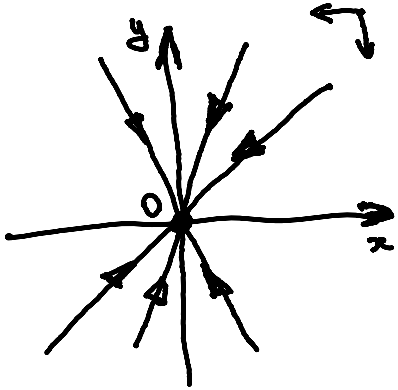
$$\det(\lambda I_2 - A) = 0$$

$$\begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 2 \end{vmatrix} = 0 \Rightarrow$$

$$(\lambda - 1)(\lambda + 2) = 0 \qquad \begin{array}{l} \lambda_1 = 1 > 0 \\ \lambda_2 = -2 < 0 \end{array}$$

$\Rightarrow (0,0)$ is a saddle point unstable.

2) $\begin{cases} x' = -x \\ y' = -y \end{cases}$



$$\frac{dx}{dy} = \frac{-x}{-y} \Rightarrow \frac{dx}{dy} = \frac{x}{y} \Rightarrow$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln y = \ln x + \ln c$$

$$\boxed{y = cx, \ c \in \mathbb{R}}$$

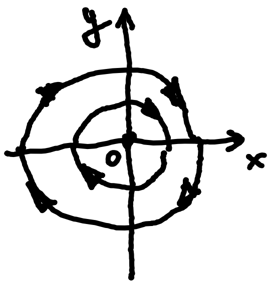
$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\det(\lambda I_2 - A) = 0$$

$$\begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda+1 \end{vmatrix} = 0 \Rightarrow (\lambda+1)^2 = 0$$

$\lambda_1 = \lambda_2 = -1 \Rightarrow (0,0)$ is a sink node (asymptotically stable)

3) $\begin{cases} x' = y \\ y' = -x \end{cases}$ $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$



$\det(\lambda I_2 - A) = 0$

$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0 \to \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$\left.\begin{array}{l} \operatorname{Re} \lambda_{1,2} = 0 \\ \lambda_{1,2} \text{ are simple} \end{array}\right\} \Rightarrow (0,0)$ is locally stable of center type.

<u>Nonlinear case</u>

$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$ $X^*(x^*, y^*)$, $f = (f_1, f_2)$

$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$ the jacobian of $f = (f_1, f_2)$

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \quad \sim \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = J_f(x^*,y^*) \begin{pmatrix} x \\ y \end{pmatrix}$$ the linearized system in $X^*(x^*,y^*)$

<u>Theorem</u> (The Stability theorem in the first approximation)

a) If $\operatorname{Re} \lambda < 0$, $\forall \lambda$ eigenvalue of $J_f(x^*,y^*)$ $\Rightarrow$

$\Rightarrow X^*$ is locally asympt. stable

b) If $\exists \lambda$ with $\operatorname{Re} \lambda > 0$, λ eigenvalue of $J_f(x^*,y^*)$ $\Rightarrow$

$\Rightarrow X^*$ is unstable.