Lecture 13

Applications of dynamical systems theory

1) Harvesting renewable resources

Suppose that a population of a certain specie is described by logistic model:

$$\begin{cases} x' = rx\left(1 - \frac{x}{K}\right) \\ x(0) = x_0 \end{cases}$$

$r, K > 0.$

r - unrestricted growth rate

K - environmental support constant. (carrying capacity const.).

<u>Problem</u>: For fixed parameters r and K determine the effect of harvesting.

a) <u>Constant rate harvesting</u>

$$\begin{cases} x' = rx\left(1 - \frac{x}{K}\right) - h \\ x(0) = x_0 \end{cases}$$

h - the constant rate of harvesting.

$h > 0$

$x' = rx\left(1-\frac{x}{K}\right) - h$ autonomous diff. eq.

Equilibrium points

$x' = f(x)$ $\quad f(x) = rx\left(1-\frac{x}{K}\right) - h$

$$f(x) = -\frac{r}{K}x^2 + rx - h$$

$f(x) = 0$ $\quad -\frac{r}{K}x^2 + rx - h = 0$

$$\Delta = r^2 - 4\cdot\left(-\frac{r}{K}\right)(-h)$$

$$= r^2 - 4\frac{rh}{K} = \underbrace{r}_{>0}\left(r - \frac{4h}{K}\right).$$

I $\quad \Delta = 0 \iff r - \frac{4h}{K} = 0 \iff \boxed{h = \frac{rK}{4}}$

$$x_1 = x_2 = \frac{-r}{2\cdot\left(-\frac{r}{K}\right)} = \frac{K}{2}.$$

the ODE has only one eq. point $x^* = \frac{K}{2} > 0$

x	$-\infty$		$\frac{K}{2}$		$+\infty$
$f(x)$	$-$ $-$ $-$		0	$-$ $-$ $-$ $-$	
		$\leftarrow$		$\leftarrow$	

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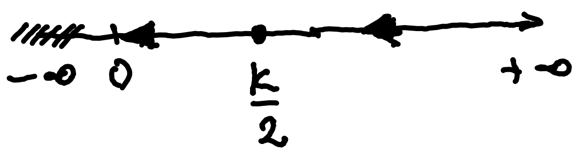
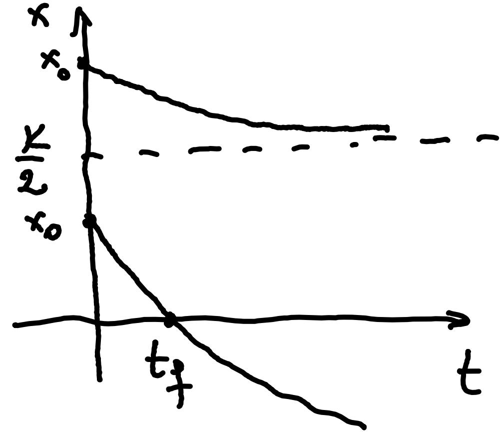




- if $x_0 > \frac{K}{2}$ then $x(t) \xrightarrow[t \to +\infty]{} \frac{K}{2}$

- if $x_0 < \frac{K}{2}$ then $x(t) \xrightarrow[t \to +\infty]{} -\infty \Rightarrow$

$\Rightarrow \exists\, t_f > 0$ such that $x(t_f) = 0$

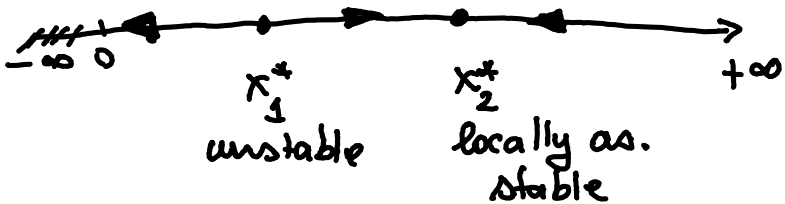
the pop. will disappear in finite time.

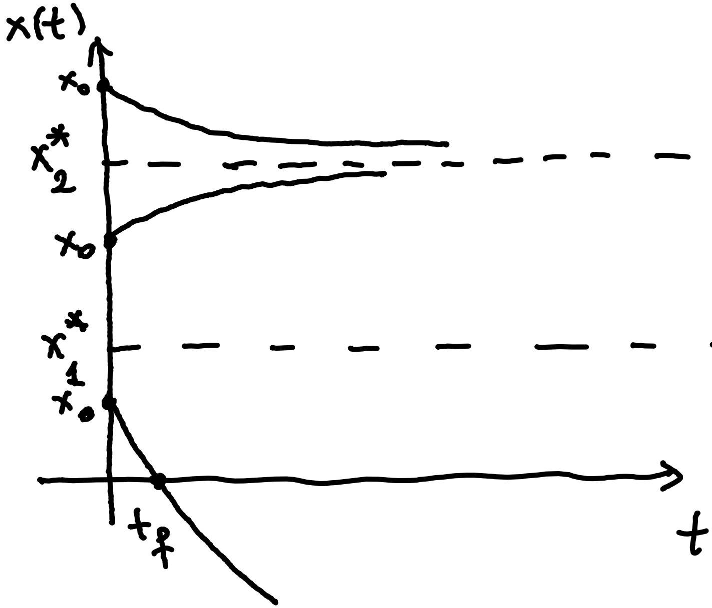
II $\Delta > 0 \iff r - \frac{4h}{K} > 0 \iff \boxed{h < \frac{rK}{4}}$

$$\Rightarrow x^*_{1,2} = \frac{-r \pm \sqrt{r^2 - \frac{4rh}{K}}}{-2\frac{r}{K}} = \frac{K}{2r}\left(r \mp \sqrt{r^2 - \frac{4rh}{K}}\right)$$

$\Rightarrow 0 < x_1^* < x_2^*$

x	$-\infty$		x_1^*		x_2^*		$+\infty$
$f(x)$		$- \; -$	0	$+ \; +$	0	$- \; - \; -$	
		$\leftarrow$		$\rightarrow$		$\leftarrow$	





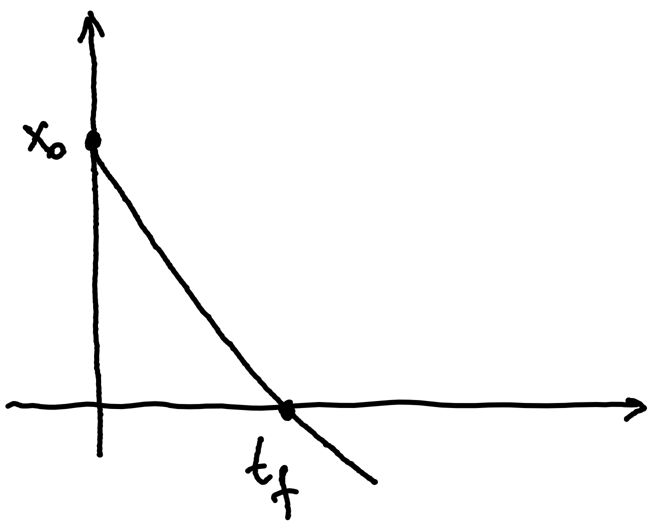
- if $x_0 > x_1^* \Rightarrow$
$\Rightarrow x(t) \xrightarrow[t\to+\infty]{} x_2^*$
the pop. tends to x_2^*

if $x_0 < x_1^* \Rightarrow$
$\Rightarrow x(t) \xrightarrow[t\to+\infty]{} -\infty \Rightarrow$
$\Rightarrow \exists\, t_f > 0$ such that $x(t_f) = 0$, the pop. will disappear in finite time.

III $\Delta < 0 \iff r - \frac{4h}{K} < 0 \iff \boxed{h > \frac{rK}{4}}$

$f(x) = 0$ has no real roots $\Rightarrow$ the ODE has no equilibrium points.

x	$-\infty$ $\qquad\qquad\qquad\qquad$ $+\infty$
$f(x)$	$-$ $\quad$ $-$ $\quad$ $-$ $\quad$ $-$ $\quad$ $-$ $\quad$ $-$ $\quad$ $-$
	$\leftarrow$ $\qquad$ $\leftarrow$



$x(t) \xrightarrow[t \to +\infty]{} -\infty$,

$\Rightarrow \exists\, t_f > 0$ such that $x(t_f) = 0$, the pop. will disappear in finite time regardless of its initial size x_0.

(the excessive harvesting)

Conclusion :

- if the harvesting const. rate h satisfies

$$0 < h \leq \frac{rK}{4}$$

then there exists a treshold value x_T such that if the initial size of the pop. x_0 is less than x_T $(x_0 < x_T)$ then the pop. is exterminated in finite time.

if the initial size of the pop. x_0 is above of x_T then the pop. tends to an equilibrium point $\bar{x}$

$$\left(\begin{array}{lll} \underline{\mathrm{I}} & x_T = \frac{K}{2} , & \bar{x} = \frac{K}{2} \\ \underline{\overline{\mathrm{II}}} & x_T = x_1^* , & \bar{x} = x_2^* \end{array} \right)$$

- if the harvesting const. rate h satisfies

$$h > \frac{rK}{4}$$

then the pop. is exterminated in finite time regardless of its initial size x_0. (the excessive harvesting).

b) <u>The proportional rate harvesting</u>

- the population is harvested with a rate proportional to the size of the population

$$\begin{cases} x' = rx\left(1-\frac{x}{K}\right) - E\cdot x \\ x(0) = x_0 \end{cases}$$

$r, K > 0$

$E > 0$

E – the effort made to harvest the given pop.

<u>Problem</u>: Study the effect of proportional rate harvesting on the growth of population.

$$x' = f(x) \qquad f(x) = rx\left(1-\frac{x}{K}\right) - E\cdot x = x\left(r\left(1-\frac{x}{K}\right)-E\right).$$

<u>Equilibrium points</u>:

$$f(x)=0 \Rightarrow x\left(r\left(1-\frac{x}{K}\right)-E\right)=0$$

$$\boxed{x_1^* = 0}$$

$$r\left(1-\frac{x}{K}\right) - E = 0$$

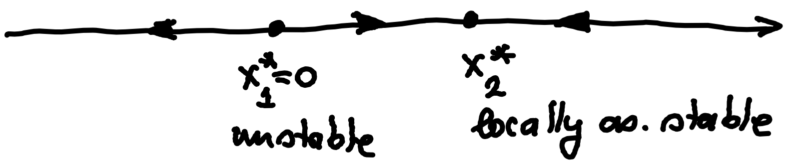
$$r - \frac{rx}{K} = E \Rightarrow \frac{rx}{K} = r - E$$

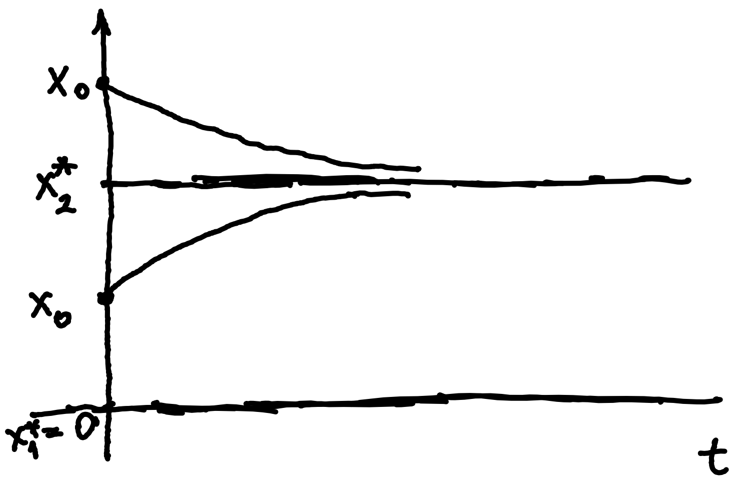
$$\boxed{x_2^* = \frac{K}{r}(r-E)}$$

$x_2^* \geq 0 \iff r \geq E$

I $\quad E < r \implies x_1^* = 0\,,\; x_2^* = \frac{K}{r}(r - E) > 0$

x	$-\infty$		$x_1^* = 0$		x_2^*	
$f(x)$	$-\;-$		0	$+\;+$	0	$-\;-$
	$\leftarrow$			$\rightarrow$		$\leftarrow$



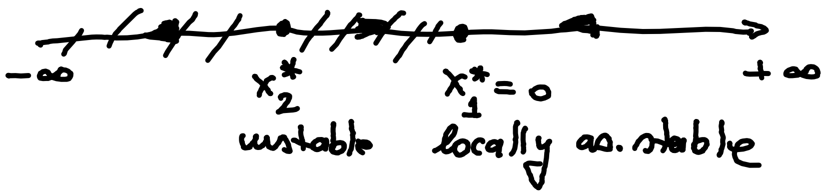


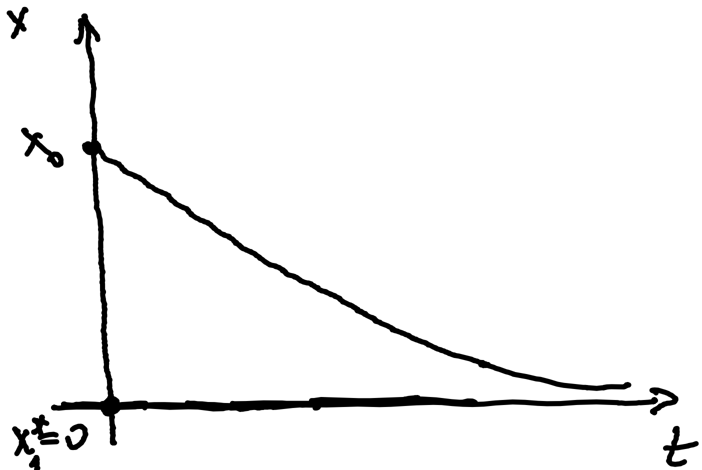
$x(t) \xrightarrow[t \to +\infty]{} x_2^* \implies$

$\implies$ the pop. tends to the eq. point x_2^*.

$\underline{\text{II}}$ $E \geq r \Rightarrow x_1^* = 0$, $x_2^* \leq 0$

x		x_2^*		$x_1^* = 0$	
$f(x)$	$-\ -$	0	$+\ +$	0	$-\ -$
	$\leftarrow$		$\rightarrow$		$\leftarrow$





$x(t) \xrightarrow[t \to \infty]{} 0$, the pop. disappears in time.

Conclusion

- if $E < r \Rightarrow x(t) \xrightarrow[t\to\infty]{} x_2^* > 0$, the pop. tends to the positive equilibrium point x_2^*.

- if $E \geq r \Rightarrow x(t) \xrightarrow[t\to+\infty]{} 0$, the pop. disappears in time

2) <u>The prey-predator model</u> (Lotka-Volterra model)

x — the prey population

y — the predator population.

x_0, y_0 — the initial size of populations.

$x(0) = x_0$

$y(0) = y_0$.

- if the predator pop. is not present in the area, we suppose that the prey pop. has unlimited food resources

$\Rightarrow$ the prey pop x will develop according to the Malthus model with positive growth rate.

$$x' = a \cdot x \quad , a > 0 .$$

- the predator pop. has no other food resources than the considered prey pop. x., so if the prey pop. is not present in the area then the predator pop. has no food so will develop according to the Malthus model but with negative growth rate

$$y' = -c.y \quad , c>0$$

- the predator population harvest prey pop. with a proportional rate to total number of possible interactions between them.

$x.y$ – the total number of possible interactions.

$$x' = ax - b.xy \quad , a,b>0$$

- the decreasing of the predator populations is limited by the harvesting with a rate proportional to the total number of possible interactions between populations

$$y' = -cy + d.xy \quad , c,d>0$$

$$\begin{cases} x' = ax - bxy \\ y' = -cy + d \cdot xy \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

$a, b, c, d > 0$ (Lotka – Volterra model).

Equilibrium points

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$$

$f_1(x,y) = ax - bxy$

$f_2(x,y) = -cy + dxy$

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases} \Rightarrow \begin{cases} x(a - by) = 0 \\ y(-c + dx) = 0 \end{cases} \Rightarrow$$

$\Rightarrow \quad x = 0 \quad$ or $\quad a - by = 0$

$x = 0 \Downarrow y \cdot (-c) = 0 \Downarrow y = 0$

$a - by = 0 \Downarrow y = \frac{a}{b} \Downarrow \underbrace{\frac{a}{b}}_{\neq 0} \cdot (-c + dx) = 0 \Rightarrow x = \frac{c}{d}$

$$\Downarrow$$

$$\begin{cases} x^* = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x^* = \frac{c}{d} \\ y^* = \frac{a}{b} \end{cases}$$

$$X_1^*(0,0) \qquad X_2^*\left(\frac{c}{d}, \frac{a}{b}\right)$$

Stability of the eq. points

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} a - by & -bx \\ d \cdot y & -c + dx \end{pmatrix}$$

$f = (f_1, f_2)$

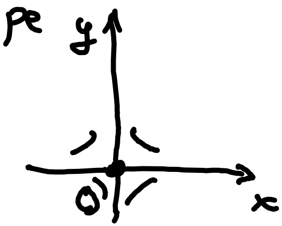
$X_1^*(0,0)$: $J_f(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

$$\det(\lambda I_2 - J_f(0,0)) = 0$$

$$\begin{vmatrix} \lambda - a & 0 \\ 0 & \lambda + c \end{vmatrix} = 0 \Rightarrow (\lambda - a)(\lambda + c) = 0$$

$$\lambda_1 = a \quad , \lambda_2 = -c .$$

$\lambda_1 = a > 0 \Rightarrow X_1^*(0,0)$ is unstable of saddle type



$\underline{X_2^*\left(\frac{a}{b}, \frac{c}{d}\right)}$:

$$J_f\left(\frac{c}{d}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{d.a}{b} & 0 \end{pmatrix}$$

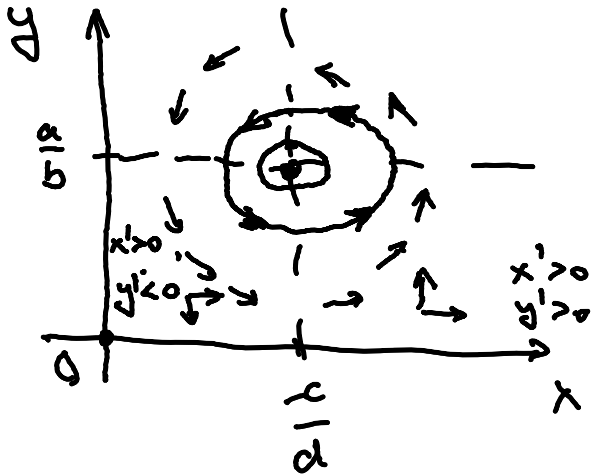
$$\det\left(\lambda I_2 - J_f\left(\frac{c}{d}, \frac{a}{b}\right)\right) = 0$$

$$\begin{vmatrix} \lambda & \frac{b.c}{d} \\ -\frac{d.a}{b} & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + a.c = 0$$

$$\lambda^2 = \underbrace{-\underbrace{a.c}_{>0}}_{<0} \Rightarrow \lambda_{1,2} = \pm i\sqrt{ac}$$

$\left.\begin{array}{l} \operatorname{Re}\lambda_{1,2} = 0 \\ \text{the system is nonlinear} \end{array}\right\} \Rightarrow$ we cannot apply the Stability Theorem in the first approximation.

The sketch of the phase portrait



$$\begin{cases} x' = x(a - by) \\ y' = y(-c + dx) \end{cases}$$

It can be proved that the orbits are closed curves. $\longrightarrow$ X_2^* is locally stable of center type.