Lecture 14

Approximating methods for IVP solutions

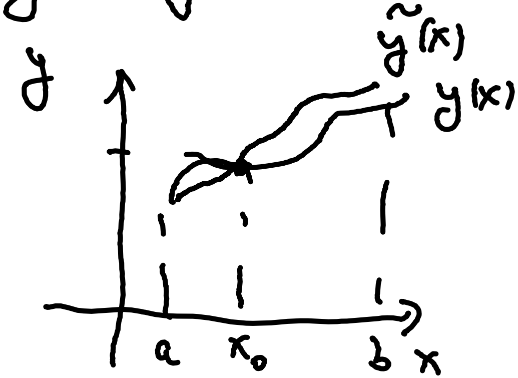
$$(1)\ \begin{cases} y' = f(x,y) \\ y(x_0) = y^0 \end{cases}\ (2) \qquad f : D_f \to \mathbb{R}. \quad x_0 \in \mathbb{R},\ y^0 \in \mathbb{R}.$$

Approximating methods

I Semi-analytical methods

$y(x)$ – exact solution of (1)+(2)

$y(x) \simeq \tilde{y}(x)$ on some I

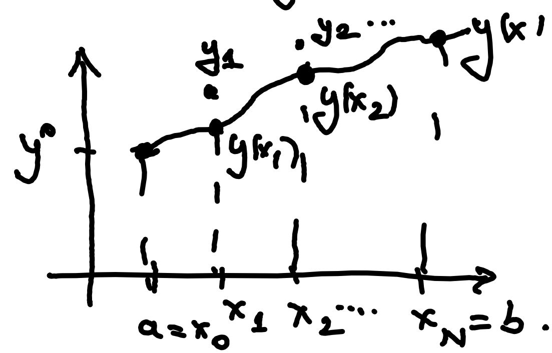


II Numerical methods

$x_0 = a \qquad I = [a,b]$

$a = x_0 < x_1 < \dots < x_N = b$

$y(x_i) \simeq y_i \ ,\ i = \overline{0,N}$



I Semi-analytical methods

1). Picard iteration method (succesive approximating sequence).

Theorem 1 (The existence and uniqueness th. in the space).

Let's consider IVP (1)+(2). Suppose that:

(i) $f \in C(I \times \mathbb{R}, \mathbb{R})$.

(ii) f is lipschitz with respect to the second variable on $I \times \mathbb{R}$.

$\exists L_f > 0$ s.t $|f(x,u) - f(x,v)| \leq L_f \cdot |u-v|$, $\forall x \in I$, $\forall u, v \in \mathbb{R}$.

Then.

a) the IVP (1)+(2) has an unique solution $y^* \in C(I, \mathbb{R})$

b) the succesive approximating sequence converges to y^* for any starting function $y_0 \in C(I, \mathbb{R})$.

(1)+(2) $\Longleftrightarrow$ (3)

(3) $$y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds.$$ the Volterra integral equation.

$A : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ $\qquad$ $(3) \Leftrightarrow y = A(y)$

$y \longmapsto A(y)$

$$A(y)(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds.$$

A is a contraction in $\left(C(I,\mathbb{R}), \|\cdot\|_\tau\right)$

the Bielecki norm.

succesive approximating sequence:

$y_0 \in C(I,\mathbb{R}) \Rightarrow y_{n+1} = A^n(y_0)$ $\qquad$ $A^n = \underbrace{A \circ A \circ \ldots \circ A}_{n \text{ times}}.$

$$\boxed{y_{n+1} = A(y_n)} = A^n(y_0)$$

$$\boxed{y_{n+1}(x) = y^0 + \int_{x_0}^{x} f(s, y_n(s))\,ds}$$

$y_0 \in C(I,\mathbb{R})$ starting function.

succesive approximating sequence (Picard iteration).

$y(x) \simeq y_n(x)$

Theorem 2 (The existence and uniqueness th. in the ball $\overline{B}(y^0, b)$)

Let's consider IVP (1)+(2). Suppose that:

$f : D_f \to \mathbb{R}$, $D_f \subseteq \mathbb{R}^2$ domain.

$\overline{D} = [x_0 - a, x_0 + a] \times [y^0 - b, y^0 + b] \subseteq D_f$, $a, b > 0$

(i) $f \in C(\overline{D}, \mathbb{R})$

(ii) f is locally lipschitz on D_f. (f is lipschitz on any compact set $K \subseteq D_f$).

Then:

a) IVP has an unique solution $y^* \in C([x_0 - h, x_0 + h], [y^0 - b, y^0 + b])$ where $h = \min\{a, \frac{b}{M_f}\}$, $M_f = \max_{\overline{D}} |f(x,y)|$.

b) the succesive aproximating sequence converges to the unique sol y^* for any starting function $y_0 \in C([x_0 - h, x_0 + h], [y^0 - b, y^0 + b])$.

Examples.

1) $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ $y^*(x) = e^x$ is the exact sol.

$x_0 = 0$, $y^0 = 1$

$f(x,y) = y$.

$\left| \frac{\partial f}{\partial y}(x,y) \right| = |1| = 1 < +\infty \Rightarrow f$ is lipschitz with respect to y on $\mathbb{R} \times \mathbb{R}$.

$0 \in I = [-a, a]$ $a > 0$

f is lipsch. with respect to y on $[-a,a] \times \mathbb{R}$.

$\xrightarrow{\text{Th 1.}}$ (y_n) the successive app. sequence converges to $y^*(x) = e^x$ for any starting function $y_0 \in C([-a,a], \mathbb{R})$.

$$y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds = 1 + \int_0^x y(s)\,ds.$$

$$\boxed{y(x) = 1 + \int_0^x y(s)\,ds}$$ the equiv. Volterra integral eq.

successive approximating sequence:

$$y_{n+1}(x) = 1 + \int_0^x y_n(s)\,ds,$$ with starting function $y_0 \in C([-a,a], \mathbb{R})$.

let's take as a starting function

$$y_0(x) \equiv 1$$

$$\Rightarrow \quad y_1(x) = 1 + \int_0^x y_0(s)\,ds = 1 + \int_0^x 1\,ds = 1 + s\Big|_0^x = 1 + x.$$

$$y_2(x) = 1 + \int_0^x y_1(s)\,ds = 1 + \int_0^x (1+s)\,ds =$$

$$= 1 + s\Big|_0^x + \frac{s^2}{2}\Big|_0^x = 1 + x + \frac{x^2}{2}$$

$$y_3(x) = 1 + \int_0^x y_2(s)\,ds = 1 + \int_0^x \left(1 + s + \frac{s^2}{2}\right) ds =$$

$$= 1 + s\Big|_0^x + \frac{s^2}{2}\Big|_0^x + \frac{s^3}{6}\Big|_0^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

by induction :

$$y_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Taylor expansion of e^x in $x_0 = 1$.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$y_n \xrightarrow[n \to \infty]{} y^* = e^x$$

we can take as an approximating solution

$\tilde{y}(x) = y_N(x)$ N fixed.

2) $\begin{cases} y' = x^2 + y^2 \\ y(0) = 0 \end{cases}$

$x_0 = 0$, $y^0 = 0$

$f(x,y) = x^2 + y^2$ $\quad f : \mathbb{R}^2 \to \mathbb{R}$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = |2y| = 2.|y| \xrightarrow[|y|\to+\infty]{} +\infty \Rightarrow$$

$\Rightarrow$ f is not lipschitz with respect to y on $\mathbb{R}^2$

$\Rightarrow$ we cannot apply Th.1.

$$D_f = [x_0-a, x_0+a]\times\mathbb{R} = [-a,a]\times\mathbb{R}, \quad a>0$$

$$\overline{D} = [x_0-a, x_0+a]\times[y^0-b, y^0+b] = [-a,a]\times[-b,b], \quad \underline{a,b>0}.$$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = 2.|y| < 2b < +\infty$$

$\Rightarrow$ f is lipschitz with respect to y on $\overline{D}$

$\underset{\text{Th.2}}{\Rightarrow}$ $\exists!\, y^* \in C([-h,h],[-b,b])$ sol. of IVP.

where $h = \min\left\{a, \frac{b}{M_f}\right\}$

$$M_f = \max_{\overline{D}} |f(x,y)| = \max_{D}\left(|x^2+y^2|\right) =$$

$$= a^2+b^2$$

the equivalent Volterra integral equation:

$$y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds = \int_0^x (s^2 + y^2(s))\,ds =$$

$$= \int_0^x s^2\,ds + \int_0^x y^2(s)\,ds = \frac{s^3}{3}\Big|_0^x + \int_0^x y^2(s)\,ds = \frac{x^3}{3} + \int_0^x y^2(s)\,ds$$

$$\boxed{y(x) = \frac{x^3}{3} + \int_0^x y^2(s)\,ds}$$

successive approximating sequence:

$$\boxed{y_{n+1}(x) = \frac{x^3}{3} + \int_0^x y_n^2(s)\,ds},$$ with the starting function $y_0 \in C([-h,h],[-b,b])$

let's choose $y_0(x) \equiv y^0 = 0$, $y_0 \in C([-h,h],[-b,b])$

$$y_1(x) = \frac{x^3}{3} + \int_0^x y_0^2(s)\,ds = \frac{x^3}{3} + \cancel{\int_0^x 0\,ds} = \frac{x^3}{3}.$$

$$y_2(x) = \frac{x^3}{3} + \int_0^x y_1^2(s)\,ds = \frac{x^3}{3} + \int_0^x \left(\frac{s^3}{3}\right)^2 ds =$$

$$= \frac{x^3}{3} + \int_0^x \frac{s^6}{9}\,ds = \frac{x^3}{3} + \frac{s^7}{63}\Big|_0^x = \frac{x^3}{3} + \frac{x^7}{63}$$

$$y_3(x) = \frac{x^3}{3} + \int_0^x y_2^2(s)\,ds = \frac{x^3}{3} + \int_0^x \left(\frac{s^3}{3} + \frac{s^7}{63}\right)^2 ds =$$

$$= \frac{x^3}{3} + \int_0^x \left(\frac{s^6}{9} + 2 \cdot \frac{s^{10}}{3.63} + \frac{s^{14}}{63^2}\right) ds$$

$$= \frac{x^3}{3} + \frac{s^7}{63}\Big|_0^x + \frac{2}{3.63.11} \cdot s^{11}\Big|_0^x + \frac{s^{15}}{63^2 \cdot 15}\Big|_0^x$$

$$y_3(x) = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2}{3.63.11} \cdot x^{11} + \frac{x^{15}}{63^2 \cdot 15}$$

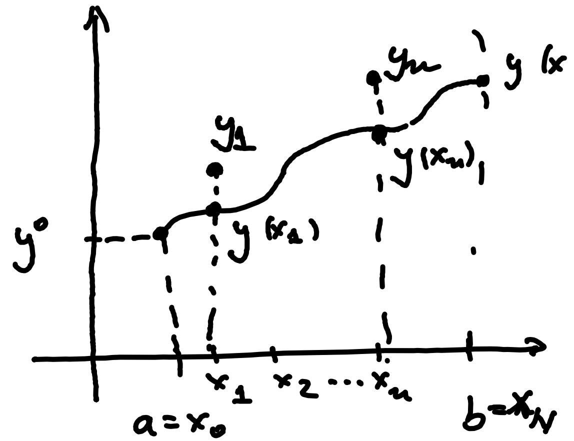
$$y^*(x) \simeq y_3(x).$$

II Numerical methods.

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y^0 \end{cases} \qquad y(x) \text{ the exact sol.}$$

$$a = x_0 < x_1 < x_2 < \dots < x_N = b.$$

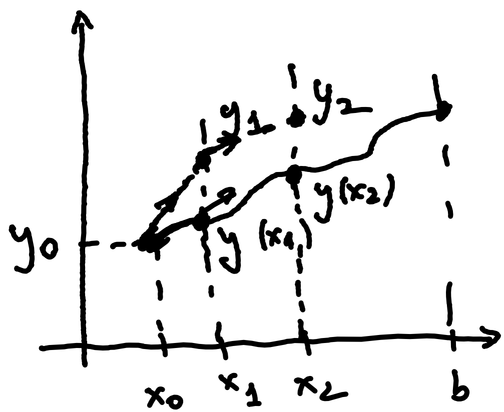
$$I = [a, b]$$



$$y(x_n) \simeq y_n$$

$$y_n = ? \text{ s.t } y_n \xrightarrow[n \to \infty]{} y(x_n)$$

1) The Euler method



$a = x_0$

$y_0 = y^0$

$(x_0, y_0) \in G_y$ – the graph of the exact sol.

$$y(x_1) \simeq y_1$$

$$y_1 = ?$$

$x = x_0 \qquad y'(x_0) = f(x_0, \underbrace{y(x_0)}_{y_0}) = f(x_0, y_0) = m$

the slope of the sol. in the point (x_0, y_0).

we approximate the sol. by the line which contain the point (x_0, y_0) with the slope $m = f(x_0, y_0)$.

$$y - y_0 = m(x - x_0)$$

(x_1, y_1) belongs to this line

$$\Rightarrow \quad y_1 - y_0 = m(x_1 - x_0)$$

$$\boxed{y_1 = y_0 + f(x_0, y_0) \cdot (x_1 - x_0)} \qquad y(x_1) \simeq y_1$$

we continue this procedure with the point (x_1, y_1)

$x = x_1 \Rightarrow y'(x_1) = f(x_1, \underbrace{y(x_1)}_{y_1}) \simeq f(x_1, y_1) = m.$

we approximate the sol. by the line which contain the point (x_1, y_1) with approximating slope $m = f(x_1, y_1)$

$$y - y_1 = m(x - x_1) \Rightarrow y - y_1 = f(x_1, y_1)(x - x_1)$$

(x_2, y_2) belongs to the line

$$\Rightarrow y_2 - y_1 = f(x_1, y_1)(x_2 - x_1)$$

$$\boxed{y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)}, \quad y(x_2) \simeq y_2$$

we continue with this procedure until we get $y(x_N) = y(b) \simeq y_N$.

(x_n, y_n) $\qquad y(x_n) \simeq y_n$

$$y'(x_n) = f(x_n, y(x_n)) \simeq f(x_n, y_n)$$

we approximate the sol. by the line which contain (x_n, y_n) with the slope $m = f(x_n, y_n)$

$$y - y_n = m(x - x_n)$$

(x_{n+1}, y_{n+1}) belongs to this line $\Rightarrow$

$$y_{n+1} - y_n = m\cdot(x_{n+1} - x_n)$$

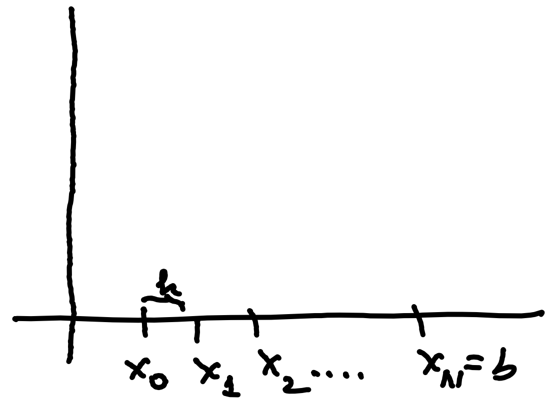
$$\Rightarrow \boxed{y_{n+1} = y_n + f(x_n, y_n)\cdot(x_{n+1} - x_n)}$$

the Euler numerical method.

if $x_{n+1} - x_n = h = \text{const}$.

$$\Rightarrow \boxed{y_{n+1} = y_n + f(x_n, y_n)\cdot h}$$

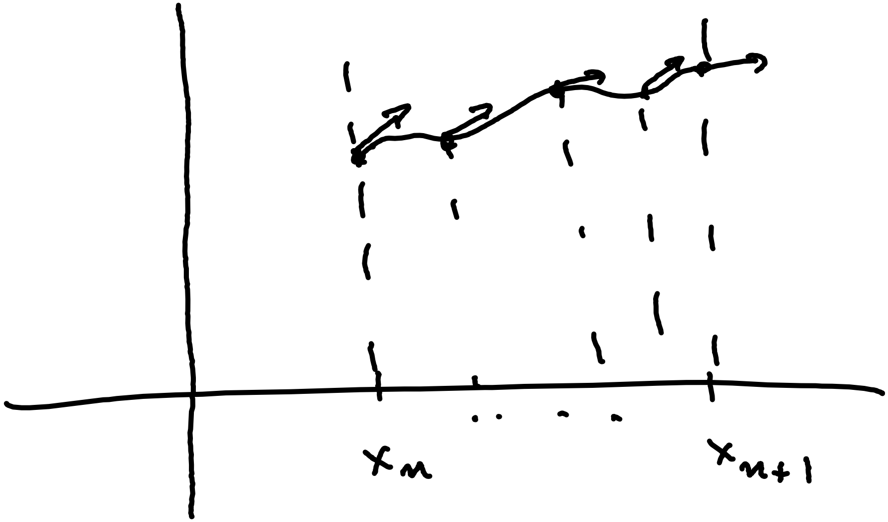
the Euler numerical method with constant step.



$$h = \frac{b-a}{N}. \rightarrow x_n = x_0 + h\cdot n \quad , \; n = \overline{0, N-1}$$

$$\boxed{y_{n+1} = y_n + f(x_n, y_n)\cdot h}$$

Runge - Kutta method.



the average slope:

$$p_n = \frac{1}{x_{n+1} - x_n} \int_{x_n}^{x_{n+1}} f(s, y(s))\, ds.$$