SEMINAR 1
ECUAȚII

First order solvable differential equations

1. ANTIDERIVATIVE PROBLEM

$y' = f(x)$ $\quad f \in C(I)$ given

$y(x) = \int_{x_0}^{x} f(s)\,ds + c, \; c \in \mathbb{R}$ the general solution

2. SEPARABLE DIFFERENTIAL EQUATIONS

General form: $y'(x) = f(x) \cdot g(y)$

$y = y(x) \Rightarrow dy = y'(x)\,dx \Rightarrow y'(x) = \frac{dy}{dx}$

$\frac{dy}{dx} = f(x) \cdot g(y) \Rightarrow \frac{dy}{g(y)} = f(x)\,dx$

$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$

$\int \frac{dy}{g(y)}$ = $G(y)$ $\qquad$ $\int f(x)\,dx$ = $F(x)$

$G(y) = F(x) + C, \; c \in \mathbb{R}$ the general solution in the implicit form

$G^{-1} \Rightarrow y(x) = G^{-1}(F(x) + c), \; c \in \mathbb{R}$ the general solution in the explicit form

Remark: If there exists $y_0 \in \mathbb{R}$ such that $g(y_0) = 0$ then $y(x) \equiv y_0$ is a solution of the separable eq. called singular solution

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = f(x) \cdot g(y)$

$g(y) = 0 \Rightarrow y(x) \equiv y_0$

Exercise 1: solve the following diff. eq.

a) $y' = 2x(1+y^2)$

b) $(x^2-1)y' + 2xy^2 = 0$

c) $xy' = y^3 + y$

d) $y' = k \cdot \frac{y}{x}$, $k \in \mathbb{R}^*$

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a) $y' = 2x(1+y^2)$

$f(x)$ $g(y)$

$g(y) = 0 \Rightarrow$

$1+y^2 = 0 \Rightarrow y^2 = -1$

$y_{1/2} = \pm i \notin \mathbb{R}$

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2x(1+y^2) \Rightarrow \frac{dy}{1+y^2} = dx \cdot 2x$

$\int \frac{dy}{1+y^2} = \int 2x\,dx \Rightarrow$ $\boxed{\operatorname{arctg} y = x^2 + c}$ general sol. in implicit form

$\boxed{y(x) = \operatorname{tg}(x^2+c)}$ general sol. in explicit form

b) $(x^2-1)y' + 2xy^2 = 0$

$(x^2-1)y' = -2xy^2$

$y' = -\frac{2xy^2}{x^2-1} = -\frac{2x}{x^2-1} \cdot y^2$

$f(x)$ $g(y)$

$g(y) = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0 \Rightarrow$ $\boxed{y(x) = 0}$ singular solution

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-2x}{x^2-1} \cdot y^2 \Rightarrow \frac{dy}{y^2} = \frac{-2x}{x^2-1}\,dx$

$\int \frac{dy}{y^2} = \int \frac{-2x}{x^2-1}\,dx$

$\frac{1}{y} = -\ln|x^2-1| + c$

$\boxed{y = -\frac{1}{\ln|x^2-1| + c}}$ general solution

c) $xy' = y^3 + y \Rightarrow y' = \frac{1}{x}(y^3 + y)$

$f(x)$ $g(y)$

$g(y) = 0 \Rightarrow y^3 + y = 0 \Rightarrow y(y^2+1) = 0 \Rightarrow y(x) = 0$ singular solution

$y' = \frac{dy}{dx} \Rightarrow \frac{1}{x}(y^3+y) = \frac{dy}{dx} \Rightarrow \frac{dy}{y^3+y} = \frac{dx}{x}$

$\int \frac{dy}{y^3+y} = \int \frac{dx}{x}$

y^2+1 $\quad$ y

$\frac{A}{y} + \frac{By+C}{y^2+1} = \frac{1}{y^3+y}$

$A(y^2+1) + y(By+C) = 1$

$Ay^2 + A + By^2 + Cy = 1$

$(A+B)y^2 + Cy + A = 1 \Rightarrow A = 1$

$C = 0$

$B = -1$

$\int \frac{1}{y} dy + \int \frac{-y}{y^2+1} dy = \ln|x| + C$

$\ln|y| - \frac{1}{2}\ln|y^2+1| = \ln|x| + c$ general sol. in implicit form

$\ln\left|\frac{y}{\sqrt{y^2+1}}\right| = \ln|x \cdot c| \Rightarrow \dots$

d) $y' = k \cdot \frac{y}{x}$, $k \in \mathbb{R}^*$

$y(x) = 0$ singular solution

$y' = \frac{dy}{dx} \Rightarrow \frac{k \cdot y}{x} = \frac{dy}{dx} \Rightarrow \frac{dy}{y} = \frac{k\,dx}{x}$

$\int \frac{dy}{y} = \int \frac{k}{x} dx \Rightarrow \ln|y| = \ln|x^k \cdot c| + c$

$\ln|y| = \ln|x|^k + \ln e^c$

$\ln|y| = \ln|x|^k \cdot e^c$

$y = x^k \cdot e^c$

$y(x) = x^k \cdot e^c$ general solution

2.

a) $\begin{cases} (1+e^x)\cdot y\cdot y' - e^x = 0 \\ y(0)=1 \end{cases}$

c) $\begin{cases} y' = K\cdot \frac{y}{x} \\ y(0)=1 \end{cases}$

b) $\begin{cases} y' \sin x - y\ln y = 0 \\ y(\frac{\pi}{2})=1 \end{cases}$

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a) $\begin{cases} (1+e^x)\cdot y\cdot y' - e^x = 0 \\ y(0)=1 \end{cases}$

$(1+e^x)\cdot y\cdot y' - e^x = 0$

$y'(1+e^x)\cdot y = e^x$

$y' = \underbrace{\frac{e^x}{1+e^x}}_{f(x)} \cdot \underbrace{\frac{1}{y}}_{g(y)}$

$g(y)=0 \Rightarrow \frac{1}{y}=0 \Rightarrow$ no singular solution

$\frac{dy}{dx} = \frac{e^x}{1+e^x}\cdot\frac{1}{y} \Rightarrow y\,dy = \frac{e^x}{1+e^x}dx$

$\int y\,dy = \int \frac{e^x}{1+e^x}dx$

$\frac{1}{2}y^2 + C = \int \frac{e^x}{e^x+1}dx$

$\frac{1}{2}y^2 = \ln|e^x+1| + c$

$y^2 = 2\ln|e^x+1| + c \Rightarrow \boxed{y_{(x)} = \pm\sqrt{2\ln|e^x+1|+c}}$ general solution

$y(0)=1 \Rightarrow \sqrt{2\ln 2 + c} = 1$

$2\ln 2 + c = 1$

$c = 1 - 2\ln 2$

$\boxed{y(x) = \sqrt{2\ln|e^x+1| + 1 - 2\ln 2}}$ the IVP solution

Ex3 *

b) $y' = (8x+2y+1)^2$

$z(x) = 8x+2y+1 \Rightarrow y = \frac{z-8x-1}{2}$

$y'(x) = \frac{1}{2}z' - 4$

$\frac{1}{2}z' - 4 = z^2 \Rightarrow z' - 8 = 2z^2$

$z' = \underbrace{2z^2 - 8}_{g(z)}$

$g(z) = 0 \Rightarrow z^2 = 4 \Rightarrow z = \pm 2 \Rightarrow \begin{cases} \frac{z-8x-1}{2} = \frac{1-8x}{2} \\ \frac{-2-8x-1}{2} = \frac{-3-8x}{2} \end{cases}$ sing. sol.

$z' = \frac{dz}{dx} \Rightarrow 2z^2 - 8 = \frac{dz}{dx} \Rightarrow 2\frac{dz}{z^2-4} = dx \Rightarrow 2\int \frac{1}{z^2-4}dz = \int 1\,dx \Rightarrow$

$\Rightarrow \frac{x}{2}\operatorname{arctg}\frac{z}{2} = x \Rightarrow z = \frac{tgx}{2} \Rightarrow y = \frac{\frac{tgx}{2} - 8x - 1}{2}$ general solution

c) $y' = \sin(x-y)$

$z(x) = x - y \Rightarrow y = x - z$

$y' = 1 - z'$

$1 - z' = \sin z$

$z' = -\sin z + 1$

$z' = \frac{dz}{dx} \Rightarrow -\sin z + 1 = \frac{dz}{dx} \Rightarrow \frac{dz}{-\sin z + 1} = dx \Rightarrow \int \frac{dz}{-\sin z + 1} = x + C$

$\int \frac{1}{-\sin z + 1}dz = \int \frac{\sin z + 1}{-\sin^2 z + 1}dz = \int \frac{\sin z + 1}{\cos^2 z}dz = \int \frac{1}{\cos^2 z}dz + \int \frac{\sin(z)}{\cos^2(x)}dz$

$= \tan z + \frac{1}{\cos z}$

$\sin^2(x) + \cos^2(x) = 1$

$1 - \sin^2(x) = \cos^2 x$

$\int \frac{\sin z}{\cos^2 z}dz = \int \frac{-1}{t^2}dt = +\frac{1}{t} = \frac{1}{\cos x}$

N: $\cos z = t$

D: $-\sin z\,dz = 1\,dt$

$\tan z + \frac{1}{\cos z} = x + C$ implicit general solution

d) $y'-1 = e^{x+2y}$

$y' = e^{x+2y} + 1$

$z(x) = x + 2y \Rightarrow y = \frac{z(x) - x}{2}$

$y'(x) = \frac{1}{2} z'(x) - \frac{1}{2}$

$\frac{1}{2} z'(x) - \frac{1}{2} = e^{z} + 1$

$z' - 1 = 2e^{z} + 2$

$z' = 2e^{z} + 1$

$\frac{dz}{dx} = 2e^{z} + 1 \Rightarrow \frac{dz}{2e^{z}+1} = dx \Rightarrow \int \frac{1}{2e^{z}+1} dz = x + C \Rightarrow$

$\Rightarrow$

3. REDUCTIBLE DIFF EQ TO THE SEPARABLE DIFF. EQ

General form: $y'(x) = f(\underbrace{ax+by+c}_{z(x)}) + d$ $\quad a,b,c,d \in \mathbb{R}$, $b \neq 0$

Subst: $z(x) = ax + by(x) + c \Rightarrow$

$\Rightarrow y(x) = \dfrac{z(x) - ax - c}{b}$ $\quad \left(\dfrac{ax+by+c-ax-c}{b} = \dfrac{by}{b}\right)$

$\Rightarrow y'(x) = \dfrac{1}{b} \cdot z'(x) - \dfrac{a}{b}$

$\dfrac{1}{b} \cdot z' - \dfrac{a}{b} = f(z) + d \quad / \cdot b$

$z' - a = b(f(z) + d)$

$z' = a + b(f(z) + d)$ separable equation

$z' = f_1(x) \cdot f_2(z)$

$f_1(x) = 1, \; f_2(z) = a + b(f(z) + d)$

$z(x) = 0 \Rightarrow y(x) = x$ singular

$z(x) = \dots\dots \Rightarrow$

$\Rightarrow y(x) = x - \dfrac{1}{\dots}$

Exercise 3: ✶

a) $y' = (y-x)^3 + 1$

b) $y' = (8x + 2y + 1)^2$

c) $y' = \sin(x - y)$

d) $y' - 1 = e^{x+2y}$

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a) $y' = \underbrace{(y-x)}^{z(x)}{}^3 + 1$

$z(x) = y - x \Rightarrow y = z(x) + x$

$z(x) = y(x) - x \quad y' = z' + 1$

$\Rightarrow z' + 1 = z^2 + 1$

$z' = z^2 \Rightarrow$ $\quad z(x) = 0$ singular solution

$\dfrac{dz}{dx} = z^2 \Rightarrow \dfrac{dz}{z^2} = \dfrac{1}{dx} \Rightarrow \int \dfrac{dz}{z^2} = \int \dfrac{1}{dx} \Rightarrow -\dfrac{1}{z} = x + c \Rightarrow$

$\Rightarrow z = -\dfrac{1}{x+c}, \; c \in \mathbb{R}$

4. HOMOGENEOUS DIFF. EQUATIONS (in the Euler sense)

General form: $y' = f(x,y)$ where f is homogeneous of 0 degree

$f(tx,ty) = f(x,y)$

$f(x,y)$ is homogeneous of degree $K \iff$

$f(tx,ty) = t^k \cdot f(x,y)$

$f(tx,ty) = f(x,y)$

$y' = f(x,y) \Rightarrow y' = F\left(\frac{y}{x}\right)$

subst $z = \frac{y}{x} \Rightarrow y = x \cdot z$

$y' = z + x \cdot z'$

$z + x \cdot z' = F(z) \Rightarrow z' = \frac{1}{x}(F(z) - z)$ reparable diff. equation

Exercise 4:

a) $2x^2 y' = x^2 + y^2 \Rightarrow y' = \frac{x^2+y^2}{2x^2} = \frac{1}{2} + \frac{1}{2}\left(\frac{y}{x}\right)^2$

subst: $z = \frac{y}{x} \Rightarrow y = x \cdot z$

$y' = z + x \cdot z'$

$\Rightarrow z + x \cdot z' = \frac{1}{2} + \frac{1}{2}(z)^2 \Rightarrow$

$\Rightarrow x \cdot z' = \frac{1}{2} + \frac{1}{2}z^2 - z$

$z' = \frac{1}{2x} \cdot (1 + z^2 - 2z)$

$z' = \underbrace{\frac{1}{2x}}_{f(x)} \cdot \underbrace{(z-1)^2}_{g(z)}$ reparable eq.

$g(z) = 0 \Rightarrow z - 1 = 0 \Rightarrow z = 1$ singular solution

$z' = \frac{dz}{dx} = \frac{1}{2x} \cdot (z-1)^2 \Rightarrow \frac{dz}{(z-1)^2} = \frac{dx}{2x} \Rightarrow$

$\Rightarrow \int \frac{dz}{(z-1)^2} dz = \int \frac{dx}{2x} dx \Rightarrow \frac{-1}{z-1} = \frac{1}{2}\ln|x| + c \Rightarrow$

$\Rightarrow z - 1 = \frac{-1}{\frac{1}{2}\ln|x| + c} \Rightarrow$

$\Rightarrow$

a) $2x^2 y' = x^2 + y^2$

b) $y' = -\frac{x+y}{y}$

c) $y' = e^{\frac{y}{x}} + \frac{y}{x}$

d) $xy' = \sqrt{x^2 - y^2 +} + y$

$\Rightarrow z = \dfrac{-1}{\frac{1}{2}\ln|x| + C} + 1 \quad , C \in \mathbb{R}$

$z(x) = 1 \Rightarrow \quad y(x) = x$ sing sol

$z(x) = \dfrac{-1}{\frac{1}{2}\ln|x| + C} \Rightarrow y(x) = \dfrac{-x}{\frac{1}{2}\ln|x| + C}$ general solution

b) $y' = \dfrac{-x+y}{y} = -\dfrac{x}{y} - 1$

$z(x) = \dfrac{y}{x} \Rightarrow y = z \cdot x$

$y' = z + x \cdot z'$

$z + x \cdot z' = -\dfrac{1}{z} - 1$

$z + x \cdot z' = -\dfrac{1}{z} - 1 \Rightarrow x \cdot z' = -\left(z + \dfrac{1}{z} + 1\right) \Rightarrow$

$z' = \dfrac{-\left(z + \frac{1}{z} + 1\right)}{x} = \underbrace{-\frac{1}{x}}_{f(x)} \underbrace{\left(z + \frac{1}{z} + 1\right)}_{g(z)}$

$g(z) = 0 \Rightarrow z + \frac{1}{z} + 1 = 0 \quad z^2 + 1 + z = 0$

$\Delta = 1 - 4 < 0 \Rightarrow \nexists$ sol $\in \mathbb{R}$

$z' = \dfrac{dz}{dx} = -\dfrac{1}{x}\left(z + \dfrac{1}{z} + 1\right) \Rightarrow \dfrac{dz}{z + \frac{1}{z} + 1} = -\dfrac{dx}{x} \Rightarrow$

$\Rightarrow \int \dfrac{dz}{z + \frac{1}{z} + 1} = \int -\dfrac{dx}{x} \Rightarrow \int \dfrac{z\,dz}{z^2 + z + 1} = \int -\dfrac{dx}{x}$

$\dfrac{z}{z^2 + z + 1} = \ldots$