SEMINAR 10

ECUATII

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

(1) $\boxed{Y' = AY + B}$ $\quad A \in C([a,b], \mathcal{M}_m(\mathbb{R}))$, $B \in C([a,b], \mathbb{R}^m)$

↳ the nonhomogeneous system

(2) $\boxed{Y' = AY}$ the homogeneous system

S_0 the sol set of (2)

S_0 is a linear subspace of the linear space $C^1([a,b], \mathbb{R}^m)$ with $\dim S_0 = m$.

$\{Y^1, \dots, Y^m\}$ basis in S_0 (the fundamental system of solutions)

$S_0 = \{c_1 Y^1 + \dots + c_m Y^m \mid c_1, \dots, c_m \in \mathbb{R}\}$

$U(x) = (Y^1 \dots Y^m)$ - the fundamental matrix of solution

$S_0 = \{U(x) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \mid c_1, \dots, c_m \in \mathbb{R}\}$

THE WRONSKIAN CRITERION

$\{Y^1, \dots, Y^m\}$ is a fundam. system of solutions $\iff$ $Y^1, \dots, Y^m$ are solutions of the system (2) and $\exists x_0 \in [a,b]$ s.t. $W(x_0; Y^1, \dots, Y^m) \neq 0$ where

$W(x; y^1, \dots, y^m) = \begin{vmatrix} y_1^1(x) & \dots & y_1^m(x) \\ \vdots & & \vdots \\ y_m^1(x) & \dots & y_m^m(x) \end{vmatrix}$ the WRONSKIAN

THE NONHOMOGENEOUS CASE

$Y' = AY + B$ the general sol

$Y = Y^0 + Y^P$

Y^0 - the gen sol. of (2)

Y^P - a particular solution of (1) which can be found using the variation of the constants method

if $U(x)$ is a fundam. matrix of sol $\Rightarrow Y^P(x) = U(x) \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_m(x) \end{pmatrix}$

1) Let's consider the system

$$\begin{cases} y_1' = y_1 \cos^2 x - (1 - \sin x \cdot \cos x) \cdot y_2 \\ y_2' = (1 + \sin x \cdot \cos x) \cdot y_1 + \sin^2 x \cdot y_2 \end{cases}$$

a) Prove that $y^1 = \begin{pmatrix} e^x \cdot \cos x \\ e^x \sin x \end{pmatrix}$ $y^2 = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}$ generate a fundam. system of solutions

b) Find the solution of the system which satisfies

$$\begin{cases} y_1(0) = 1 \\ y_2(0) = 0 \end{cases} \quad Y$$

a) $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad Y' = A \cdot Y$ where

$$A = \begin{pmatrix} y_1 \cos^2 x & -(1 - \sin \cdot \cos x) \cdot y_2 \\ 1 + \sin x \cos x & \sin^2 x \cdot y_2 \end{pmatrix}$$

$\{y^1, y^2\}$ is a fundam system of sol $\iff$ y^1, y^2 are sol. and $W(x; y^1; y^2) \neq 0$

Y^1 is a solution of the system $\iff$ $y_1(x) = e^x \cos x$, $y_2(x) = e^x \sin x$ satisfy the system equations

$y_1' = y_1 \cos^2 x - (1 - \sin x \cdot \cos x) \cdot y_2$

$e^x \cos x - \sin x\, e^x = e^x \cos x \cdot \cos^2 x - (1 - \sin x \cdot \cos x) e^x \sin x$

$e^x \cos x - \cancel{\sin x\, e^x} = e^x \cos^3 x - \cancel{e^x \sin x} + e^x \sin^2 x \cos x$ $\quad (1 - \cos^2 x)$

$e^x \cos x = e^x \cos^3 x + e^x (1 - \cos^2 x) \cos x$

$\cancel{e^x \cos x} = \cancel{e^x \cos^3 x} + \cancel{e^x \cos x} - \cancel{e^x \cos^3 x}$

$0 = 0$ TRUE

+ same for the second one!

$y_2' = (1+\sin x \cos x)\, y_1 + \sin^2 x \cdot y_2$

$e^x \sin x + e^x \cos x = (1+\sin x \cos x)\cdot e^x \cos x + \sin^2 x \cdot e^x \sin x$

$e^x \sin x + \cancel{e^x \cos x} = \cancel{e^x \cos x} + \sin x \cos^2 x\, e^x + \sin^3 x \cdot e^x$

$\cancel{e^x \sin x} = \cancel{\sin x\, e^x} - \cancel{\sin^3 x\, e^x} + \cancel{\sin^3 x\, e^x}$ T

$\Rightarrow y^1$ is a solution of the system

Y^2 is a sol of the system $\Leftrightarrow$ $y_1(x) = -\sin x$, $y_2(x) = \cos x$ satisfy the system eq.

$y_1' = y_1 \cos^2 x - (1-\sin x \cdot \cos x)\, y_2$

$-\cos x = -\sin x(\cos^2 x) - (1-\sin x \cos x)\cos x$

$\cancel{-\cos x} = \cancel{-\sin x \cos^2 x} - \cancel{\cos x} + \cancel{\sin x \cos^2 x}$ T

$y_2' = (1+\cos x \sin x)\, y_1 + \sin^2 x\, y_2$

$-\sin x = (1+\cos x \sin x) - \sin x + \sin^2 x \cdot \cos x$

$\cancel{-\sin x} = \cancel{-\sin x} - \cancel{\cos x \sin^2 x} + \cancel{\cos x \sin^2 x}$ T

$\Rightarrow y^2$ is a sol of the system

$$W(x; y^1; y^2) = \begin{vmatrix} e^x \cos x & -\sin x \\ e^x \sin x & \cos x \end{vmatrix} = e^x \cos^2 x + e^x \sin^2 x = e^x(\underbrace{\cos^2 x + \sin^2 x}_{1}) = e^x \neq 0$$

$\Rightarrow \{y^1, y^2\}$ are linearly independent $\Rightarrow \{y^1, y^2\}$ is the fundamental system of solutions

b)

$$\begin{cases} y_1(0) = 1 \\ y_2(0) = 0 \end{cases}$$

$\{y^1; y^2\}$ is a fundamental system of sol

$U(x) = (y^1\ y^2) = \begin{pmatrix} e^x \cos x & -\sin x \\ e^x \sin x & \cos x \end{pmatrix}$ is a fundam matrix of sol

$\Rightarrow$ the gen sol of the system:

$$y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = U(x)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^x \cos x & -\sin x \\ e^x \sin x & \cos x \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} c_1 e^x \cos x - c_2 \sin x \\ c_1 e^x \sin x + c_2 \cos x \end{pmatrix}$$

$$y_1(x) = c_1 e^x \cos x - c_2 \sin x$$

$$y_2(x) = c_1 e^x \sin x + c_2 \cos x \qquad c_1, c_2 \in \mathbb{R}$$

$$\begin{cases} y_1(0) = 1 \Rightarrow c_1 = 1 \\ y_2(0) = 0 \Rightarrow c_2 = 0 \end{cases} \Rightarrow \boxed{\begin{cases} y_1(x) = e^x \cos x \\ y_2(x) = e^x \sin x \end{cases}}$$

2) Find the IVP solution:

$$\begin{cases} y' = -e^x y - z \cdot \cos x \\ z' = -y - (1+x^4) z \\ y(0) = 0 \\ z(0) = 0 \end{cases} \qquad Y = \begin{pmatrix} y \\ z \end{pmatrix}$$

$$Y' = A \cdot Y, \quad A = \begin{pmatrix} -e^x & -\cos x \\ -1 & -(1+x^4) \end{pmatrix}$$

$Y \equiv 0 \Rightarrow \begin{cases} y(x) = 0 \\ z(x) = 0 \end{cases}$ is a solution of the given IVP

$\exists!$ Th

$\Rightarrow Y \equiv 0$ is the only sol of the IVP

(any IVP attached to a linear system has an unique sol)

3. Let's consider the system:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2 - x^2 \end{cases}$$

a) prove that $y^1 = \begin{pmatrix} e^x \\ e^x \end{pmatrix}$ $y^2 = \begin{pmatrix} e^{-x} \\ -e^{-x} \end{pmatrix}$ generate a f.s.s.

b) find the general sol of the system

$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow y' = AY + B$ where

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B = \begin{pmatrix} 0 \\ 2 - x^2 \end{pmatrix}$ a nonhomogeneous system

the general solution : $Y = Y^0 + Y^p$

Y^0 - the gen sol. of the homogeneous system

Y^p - a particular solution of the nonhomogeneous syst.

Y^1 is a sol of the homog. system $Y' = AY$ if

$$\begin{cases} y_1(x) = e^x \\ y_2(x) = e^x \end{cases}$$ verify the eq. of the homog. syst:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1 \end{cases} \Rightarrow \begin{cases} e^x = e^x \\ e^x = e^x \end{cases} \quad T$$

Y^2 is a sol of the homog. system if

$$\begin{cases} y_1'(x) = y_2(x) \\ y_2'(x) = y_1(x) \end{cases} \Rightarrow \begin{cases} (e^{-x})' = -e^{-x} \\ e^{-x} = e^{-x} \end{cases} \quad T$$

$\Rightarrow y^1, y^2$ are sol. of the fundamental system

$$W(x; y^1, y^2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = 2 \neq 0 \Rightarrow$$

$\{y^1, y^2\}$ l.i.

$\Rightarrow \{y^1, y^2\}$ is a f. system of solutions b)

$U(x) = (y^1\ y^2) = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix}$ the fundamental matrix of sol.

$\Rightarrow Y^0 = U(x)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},\ c_1, c_2 \in \mathbb{R}$

$Y^p = ?$ a particular solution of $Y' = AY + B$

$$Y^p(x) = U(x)\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix}\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = ?$$

$$Y^p(x) = \begin{pmatrix} y_p^1(x) \\ y_p^2(x) \end{pmatrix} = \begin{pmatrix} e^x\varphi_1(x) + e^{-x}\varphi_2(x) \\ e^x\varphi_1(x) - e^{-x}\varphi_2(x) \end{pmatrix}$$

$$\begin{cases} (y_p^1)'(x) = (y_p^2)(x) \\ (y_p^2)'(x) = (y_p^1)(x) + 2 - x^2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cancel{e^x\varphi_1} + e^x\varphi_1' - \cancel{e^{-x}\varphi_2} + e^{-x}\varphi_2' = \cancel{e^x\varphi_1} - \cancel{e^{-x}\varphi_2} \\ \cancel{e^x\varphi_1} + e^x\varphi_1' + \cancel{e^{-x}\varphi_2} - e^{-x}\varphi_2' = \cancel{e^x\varphi_1} + \cancel{e^{-x}\varphi_2} + 2 - x^2 \end{cases}$$

$$\begin{cases} e^x\varphi_1' + e^{-x}\varphi_2' = 0 \\ e^x\varphi_1' - e^{-x}\varphi_2' = 2 - x^2 \end{cases}$$

$$2e^x\varphi_1' = 2 - x^2$$

$$\varphi_1' = \frac{2 - x^2}{2e^x} = \frac{1}{2}(2 - x^2)e^{-x}$$

$$\Rightarrow \varphi_2' = -\frac{1}{2}(2 - x^2)e^x$$

$$\varphi_1(x) = \int \frac{1}{2}(2 - x^2)e^{-x}\,dx = -e^{-x} - \frac{1}{2}\int x^2 \cdot e^{-x}\,dx = -e^{-x} + \frac{1}{2}\int \left(x^2 \cdot \overbrace{-e^{-x}}^{(e^{-x})'}\right)dx =$$

$$= -e^{-x} + \frac{1}{2}x^2 e^{-x} - \frac{1}{2}\int 2x \cdot e^{-x}\,dx = -e^{-x} + \frac{1}{2}x^2 e^{-x} + \int x\underbrace{(-e^{-x})}_{(e^{-x})'}\,dx =$$

$$= -e^{-x} + \frac{1}{2}x^2 e^{-x} + xe^{-x} - \int e^{-x}\,dx = \cancel{-e^{-x}} + \frac{1}{2}x^2 e^{-x} + xe^{-x} \cancel{+ e^{-x}}$$

$$\Rightarrow \varphi_1(x) = \left(\frac{x^2}{2} + x\right)e^{-x}$$

$$\varphi_2'(x) = -e^x + \frac{x^2}{2} e^x$$

$$\Rightarrow \varphi_2(x) = \int \left(-e^x + \frac{x^2}{2} e^x\right) dx = -e^x + \int \frac{x^2}{2} e^x dx =$$

$$= -e^x + \frac{x^2}{2} e^x - \int \frac{1}{\cancel{2}} \cancel{2}x \cdot e^x dx = -e^x + \frac{x^2}{2} e^x - \int x e^x dx =$$

$$= -e^x + \frac{x^2}{2} e^x - x e^x + \int e^x dx = \cancel{-e^x} + \frac{x^2}{2} e^x - x e^x \cancel{+ e^x}$$

$$\Rightarrow \boxed{\varphi_2(x) = \left(\frac{x^2}{2} - x\right) e^x}$$

$$Y^p(x) = \begin{pmatrix} y_p^1(x) \\ y_p^2(x) \end{pmatrix} = \begin{pmatrix} e^x \varphi_1(x) + e^{-x} \varphi_2(x) \\ e^x \varphi_1(x) - e^{-x} \varphi_2(x) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{x^2}{2} + \cancel{x} + \frac{x^2}{2} \cancel{-x} \\ \cancel{\frac{x^2}{2}} + x - \cancel{\frac{x^2}{2}} + x \end{pmatrix} = \begin{pmatrix} x^2 \\ 2x \end{pmatrix} \Rightarrow \begin{cases} y_p^1(x) = x^2 \\ y_p^2(x) = 2x \end{cases}$$

the general solution of the system

$$Y = Y^0 + Y^p$$

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} x^2 \\ 2x \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1(x) = c_1 e^x + c_2 e^{-x} + x^2 \\ y_2(x) = c_1 e^x - c_2 e^{-x} + 2x \end{cases} \quad c_1, c_2 \in \mathbb{R}$$