

LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

$$Y' = A \cdot Y + B$$

$$A \in M_m(\mathbb{R}), B \in C(I, \mathbb{R}^m)$$

the general solution $Y = Y^0 + Y^P$

Y^0 - the general solution of the homog. syst. $Y' = AY$

Y^P - is a particular sol of the non homog syst.

$Y' = AY + B$, which can be found using the variation of the constants method

U - fundamental matrix of solutions

$U = (Y^1 \dots Y^m)$ $Y^1, \dots, Y^m$ sol. linearly independent of the syst. $Y' = A \cdot Y$

$$Y^0 = U \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}, c_1, \dots, c_m \in \mathbb{R}$$

The homogeneous case

$$Y' = A \cdot Y$$

I The reduction method to a m-order linear diff. eq. with constant coefficients

$$1) \begin{cases} y_1' = y_1 - 5y_2 \\ y_2' = 2y_1 - y_2 \end{cases}$$

We derivate w.r.t x

$$y_1' = y_1 - 5y_2$$

$$y_1'' = (y_1' - 5y_2') = y_1 - 5y_2 + (2y_1 - y_2)(-5) = y_1 - 5y_2 - 10y_1 + 5y_2 = -9y_1$$

$$y_1 = y_1(x)$$

$$y_2 = y_2(x)$$

$$y_1'' + 9y_1 = 0$$

the second order linear homog. eq with constant coefficients

$$\pi^2 + 9\pi^0 = 0 \Rightarrow \pi^2 + 9 = 0 \Rightarrow \pi = \pm i3$$

$$\alpha = 0 \quad \beta = 3$$

$$y_{11}(x) = e^{\alpha x} \cdot \cos \beta x = \cos 3x$$

$$y_{12}(x) = e^{\alpha x} \cdot \sin \beta x = \sin 3x$$

fundamental system of solutions

$$y_1(x) = c_1 \cos 3x + c_2 \sin 3x \quad c_1, c_2 \in \mathbb{R}$$

$$y_1' = y_1 - 5y_2 \Rightarrow y_2 = y_1 - y_1'$$

$$\begin{aligned} \Rightarrow y_2 &= \frac{1}{5}(y_1 - y_1') = \frac{1}{5} [c_1 \cos 3x + c_2 \sin 3x - (c_1 \cos 3x + c_2 \sin 3x)'] = \\ &= \frac{1}{5} [c_1 \cos 3x + c_2 \sin 3x + c_1 \sin 3x - c_2 \cos 3x] = \\ &= \frac{1}{5} [c_1 (\cos 3x + \sin 3x) + c_2 (\sin 3x - \cos 3x)] \end{aligned}$$

$$\begin{cases} y_1(x) = c_1 \cos 3x + c_2 \sin 3x \\ y_2(x) = \frac{1}{5} (c_1 (\cos 3x + \sin 3x) + c_2 (\sin 3x - \cos 3x)) \end{cases}$$

$$U(x) = \begin{pmatrix} \overset{c_1}{\cos 3x} & \overset{c_2}{\sin 3x} \\ \frac{1}{5}(\cos 3x + \sin 3x) & \frac{1}{5}(\sin 3x - \cos 3x) \end{pmatrix}$$

a fundamental matrix of solutions for the system

$$Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = U(x) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad c_1, c_2 \in \mathbb{R}$$

$$2. \begin{cases} y_1' = -4y_1 + 2y_2 + 5y_3 \\ y_2' = 6y_1 - y_2 - 6y_3 \\ y_3' = -8y_1 + 3y_2 + 9y_3 \end{cases}$$

$$y_1' = -4y_1 + 2y_2 + 5y_3$$

we derivate w.r.t x

$$\begin{aligned} y_1'' &= -4y_1' + 2y_2' + 5y_3' = -4(-4y_1 + 2y_2 + 5y_3) + 2(6y_1 - y_2 - 6y_3) + 5(-8y_1 + 3y_2 + 9y_3) \\ &= 16y_1 - 8y_2 - 20y_3 + 12y_1 - 2y_2 - 12y_3 - 40y_1 + 15y_2 + 45y_3 = \\ &= -12y_1 + 5y_2 + 13y_3 \end{aligned}$$

$$y_1''' = -12y_1' + 5y_2' + 13y_3' = 48y_1 - 24y_2 - 60y_3 + 30y_1 - 5y_2 - 30y_3 - 104y_1 + 39y_2 + 117y_3$$

$$y_1''' = -26y_1 + 10y_2 + 27y_3$$

we need to express y_2, y_3 in terms of y_1 (or y_1' or y_1'')

$$\begin{cases} y_1' = -4y_1 + 2y_2 + 5y_3 \\ y_1'' = -12y_1 + 5y_2 + 13y_3 \\ y_1''' = -26y_1 + 10y_2 + 27y_3 \end{cases} \Rightarrow \begin{cases} 2y_2 + 5y_3 = y_1' + 4y_1 \quad / (-5) \\ 5y_2 + 13y_3 = y_1'' + 12y_1 \quad / \cdot 2 \end{cases}$$

from the first 2 equations we get y_3 in terms of y_1 by subtracting multiplying them adding the functions r.t. y_2 disappears

$$\begin{aligned} -10y_2 - 25y_3 &= -5y_1' - 20y_1 \\ 10y_2 + 26y_3 &= 2y_1'' + 24y_1 \\ \hline y_3 &= 2y_1'' - 5y_1' + 4y_1 \end{aligned}$$

$$y_3 = 2y_1'' - 5y_1' + 4y_1 \quad | \cdot 27$$

montage substitution for bottom eq. II

$$2y_2 + 5y_3 = y_1' + 4y_1 \Rightarrow y_2 = \frac{1}{2}(y_1' + 4y_1 + \underline{5y_3}) =$$

we get y_2 by replacing in the first equation y_3

$$= \frac{1}{2}(y_1' + 4y_1 - 10y_1'' + 25y_1' - 20y_1) =$$

$$y_2 = \frac{1}{2}(-10y_1'' + 26y_1' - 16y_1)$$

$$\Rightarrow y_2 = -5y_1'' + 13y_1' - 8y_1 \quad | / 10$$

$$y_1''' = -26y_1' + 10y_2 + 27y_3$$

$$y_1''' = -26y_1' - 50y_1'' + 130y_1' - 80y_1 + 54y_1'' - 155y_1' + 108y_1$$

$$y_1''' = 4y_1'' - 5y_1' + 2y_1$$

$$\Rightarrow y_1''' - 4y_1'' + 5y_1' - 2y_1 = 0$$

after getting y_1, y_2, y_3 in terms of y_1 we replace them in the last equation to get a linear equation

Solving the equation

$$\pi^3 - 4\pi^2 + 5\pi - 2 = 0$$

$$\pi^3 - 2\pi^2 - 2\pi^2 + 4\pi + (\pi - 2) = \pi^2(\pi - 2) - 2\pi(\pi - 2) + \pi - 2 = 0$$

$$\Rightarrow (\pi - 2)(\pi^2 - 2\pi + 1) = 0 \Rightarrow$$

$$\Rightarrow (\pi - 2)(\pi - 1)^2 = 0$$

$$\pi = 2 \rightarrow e^{2x}$$

$$\pi_2 = \pi_3 \Rightarrow \begin{cases} e^x \\ x \cdot e^x \end{cases}$$

$$\Rightarrow y_1(x) = c_1 e^{2x} + c_2 e^x + c_3 x e^x$$

$$c_1, c_2, c_3 \in \mathbb{R}$$

$$y_2 = -5y_1'' + 13y_1' - 8y_1 = \dots$$

$$y_3 = 2y_1'' - 5y_1' + 4y_1 = \dots$$

Homework

II. The method of characteristic equation

$$Y' = AY, \quad A \in \mathcal{M}_m(\mathbb{R})$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

we look for sol. of the form

$$Y = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \vdots \\ \alpha_m e^{\lambda x} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow \lambda$ is a solution of the ~~char~~ equation

$$\boxed{\det(\lambda I_m - A) = 0} \quad \text{the charact. eq.}$$

and $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$ is a nonzero sol. of the system:

$$(\lambda I_m - A) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = 0$$

$$1) \begin{cases} y_1' = y_1 - 5y_2 \\ y_2' = 2y_1 - y_2 \end{cases}$$

$$A = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix}$$

$$\det(\lambda I_2 - A) = 0$$

$$\begin{vmatrix} \lambda - 1 & 5 \\ -2 & \lambda + 1 \end{vmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda + 1) + 10 = 0$$
$$\lambda^2 - 1 + 10 = 0$$
$$\lambda^2 = -9$$
$$\lambda = \pm 3i$$

$$(\lambda I_2 - A) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

we find a nonzero sol in $\mathbb{C}$

$$\lambda = 3i \Rightarrow \begin{pmatrix} 3i - 1 & 5 \\ -2 & 3i + 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (-1 + 3i)\alpha_1 + 5\alpha_2 = 0 \\ -2\alpha_1 + (1 + 3i)\alpha_2 = 0 \end{cases}$$

$$-2\alpha_1 + (1 + 3i)\alpha_2 = 0 \Rightarrow \alpha_2 = \frac{2\alpha_1}{1 + 3i}$$

$$\text{we choose } \alpha_1 = 1 \Rightarrow \alpha_2 = \frac{2}{1 + 3i} = \frac{2 - 6i}{10}$$

$$\boxed{\alpha_1 = 1} \quad \boxed{\alpha_2 = \frac{1}{5} - \frac{3}{5}i}$$

$$e^{\alpha x + i\beta x} = e^{\alpha x} (\cos \beta x + i \sin \beta x)$$

$$\Rightarrow Y = \begin{pmatrix} \alpha_1 e^{2x} \\ \alpha_2 e^{2x} \end{pmatrix} = \begin{pmatrix} e^{2ix} \\ \left(\frac{1}{5} - \frac{3}{5}i\right) e^{2ix} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos 3x + i \sin 3x \\ \left(\frac{1}{5} - \frac{3}{5}i\right)(\cos 3x + i \sin 3x) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos 3x \\ \frac{1}{5} \cos 3x + \frac{3}{5} \sin 3x \end{pmatrix}}_{Y^1} + i \underbrace{\begin{pmatrix} \sin 3x \\ -\frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \end{pmatrix}}_{Y^2}$$

$$U(x) = \begin{pmatrix} \cos 3x & \sin 3x \\ \frac{1}{5} \cos 3x + \frac{3}{5} \sin 3x & -\frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \end{pmatrix} \quad \text{fundam. matrix of sol}$$

The gen. sol:

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = U(x) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1(x) = c_1 \cos 3x + c_2 \sin 3x \\ y_2(x) = c_1 \left(\frac{1}{5} \cos 3x + \frac{3}{5} \sin 3x \right) + c_2 \left(-\frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \right) \end{cases} \quad c_1, c_2 \in \mathbb{R}$$

$$2) \begin{cases} y_1' = y_2 + y_3 \\ y_2' = 3y_1 + y_3 \\ y_3' = 3y_1 + y_2 \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

$$\det(\lambda I_3 - A) = 0$$

$$\begin{vmatrix} \lambda & -1 & -1 \\ -3 & \lambda & -1 \\ -3 & 1 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 7\lambda - 6 = 0$$

$$(\lambda + 1)(\lambda + 2)(\lambda - 3) = 0$$

$$\lambda_1 = -1 \quad \lambda_2 = -2 \quad \lambda_3 = 3$$

$$Y = \begin{pmatrix} \alpha_1 e^{2x} \\ \alpha_2 e^{2x} \\ \alpha_3 e^{2x} \end{pmatrix} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(\lambda I_3 - A) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_1 = -1 \quad (2I_3 - A) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 \\ -3 & -1 & -1 \\ -3 & -1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 - \alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} -\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$$

$$\boxed{\alpha_3 = 1} \Rightarrow \begin{cases} -\alpha_1 - \alpha_2 = 1 \\ 3\alpha_1 + \alpha_2 = -1 \end{cases}$$

$$2\alpha_1 = 0 \Rightarrow \boxed{\alpha_1 = 0}$$

$$\Rightarrow \boxed{\alpha_2 = -1}$$

$$y' = \begin{pmatrix} 0 \\ -e^{-x} \\ e^{-x} \end{pmatrix}$$

$$\lambda_2 = -2 \quad (2I_3 - A) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & -1 & -1 \\ -3 & -2 & -1 \\ -3 & -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - 2\alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 - 2\alpha_3 = 0 \end{cases}$$

$$\boxed{\alpha_1 = 1} \Rightarrow \begin{cases} -\alpha_2 - \alpha_3 = 2 \\ -2\alpha_2 - \alpha_3 = 3 \end{cases}$$

$$\boxed{\alpha_2 = -1}$$

$$\Rightarrow \boxed{\alpha_3 = 1}$$

$$\Rightarrow y_2 = \begin{pmatrix} e^{2x} \\ -e^{-2x} \\ e^{2x} \end{pmatrix}$$

$$\lambda_3 = 3 \quad (2I_3 - A) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 3 & -1 & -1 \\ -3 & 3 & -1 \\ -3 & -1 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{cases} 3\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 + 3\alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 + 3\alpha_3 = 0 \end{cases}$$

$$\begin{cases} -\alpha_2 - \alpha_3 = -3\alpha_1 \\ 3\alpha_2 - \alpha_3 = 3\alpha_1 \end{cases}$$

$$\text{for } \alpha_1 = 1 \Rightarrow \begin{cases} -\alpha_2 - \alpha_3 = -3 \\ 3\alpha_2 - \alpha_3 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_2 + \alpha_3 = 3 \\ 3\alpha_2 - \alpha_3 = 3 \end{cases}$$

$$4\alpha_2 = 6 \Rightarrow \boxed{\alpha_2 = \frac{3}{2}}$$

$$\Rightarrow \alpha_3 = 3 - \alpha_2 = 3 - \frac{3}{2} = \frac{3}{2}$$

$$\Rightarrow y^3 = \begin{pmatrix} e^{3x} \\ \frac{3}{2} e^{3x} \\ \frac{3}{2} e^{3x} \end{pmatrix}$$

$$U = (y^1, y^2, y^3) = \begin{pmatrix} 0 & e^{-2x} & e^{3x} \\ -e^{-x} & -e^{-2x} & \frac{3}{2} e^{3x} \\ e^{-x} & -3e^{-2x} & \frac{3}{2} e^{3x} \end{pmatrix}$$

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = U \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1(x) = c_2 e^{-2x} + c_3 e^{3x} \\ y_2(x) = -c_1 e^{-x} - c_2 e^{-2x} + \frac{3}{2} c_3 e^{3x} \\ y_3(x) = c_1 e^{-x} - 3c_2 e^{-2x} + \frac{3}{2} c_3 e^{3x} \end{cases}$$

$$c_1, c_2, c_3 \in \mathbb{R}$$

the general solution