SEMINAR 12

ECUAȚII

DYNAMICAL SYSTEMS GENERATED BY AUTONOMOUS SCALAR DIFFERENTIAL EQUATIONS

$x' = f(x)$ $\quad x = x(t)$ $\quad , f \in C^1$

the flow = the saturated solution of the IVP

$$\begin{cases} x' = f(x) \\ x(0) = \eta \quad , \eta \in \mathbb{R} \end{cases} \Rightarrow x(t, \eta)$$

$x(\cdot, \eta) : I_\eta \to \mathbb{R}$, I_η - maximal

$I_\eta = (\alpha_\eta, \beta_\eta)$, $0 \in I_\eta$

if $I_\eta = \mathbb{R} \Rightarrow W = \mathbb{R}^2$

$W = \{ I_\eta \times \mathbb{R} \mid \eta \in \mathbb{R} \}$

$\varphi : W \to \mathbb{R}$

$\varphi(t, \eta) = x(t, \eta)$

Properties of the flow:

1. $\varphi(0, \eta) = \eta$
2. $\varphi(t+s, \eta) = \varphi(t, \varphi(s, \eta))$
3. φ is cont.

Orbits

$\gamma^+(\eta) = \bigcup_{t \in [0, \alpha_\eta)} \varphi(t, \eta)$ - the positive orbit of η

$\gamma^-(\eta) = \bigcup_{t \in (\alpha_\eta, 0]} \varphi(t, \eta)$ - the negative orbit of η

$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta)$ - the orbit of η

PHASE PORTRAIT (the phase line): the collection of all orbits with the describing direction

1. Let's consider the eq $x' = x+1$

a) find the generated flow

b) find the orbits for $\eta = -1, \eta = 0, \eta = -2$

c) find the phase portrait

Remainder:
flow = saturated sol of:
$\begin{cases} x' = f(x) \\ x(0) = \eta \end{cases}$

a) $\begin{cases} x' = x+1 \\ x(0) = \eta \end{cases}$ $\quad \frac{dx}{dt} = x+1 \Rightarrow \int \frac{dx}{x+1} = \int dt$

$\Rightarrow \ln(x+1) = t + \ln c$

$x+1 = c \cdot e^{t}$

$\boxed{x(t) = c \cdot e^{t} - 1, c \in \mathbb{R}}$ general solution of the diff eq.

$x(0) = \eta \Rightarrow c - 1 = \eta \Rightarrow c = \eta + 1$

$\Rightarrow x(t, \eta) = (\eta+1)e^{t} - 1$

$x(\cdot, \eta) : I_\eta \to \mathbb{R}$ $\quad I_\eta$ maximal

$\Rightarrow I_\eta = \mathbb{R} \quad \forall \eta \in \mathbb{R}$

$\varphi(t, \eta) = x(t, \eta) = \boxed{(\eta+1)e^{t} - 1}$ } the flow

$\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

b) $\eta = -1 \quad \eta = 0 \quad \eta = -2$

$\gamma(\eta) = \bigcup_{t \in I_\eta} \varphi(t, \eta)$

$\gamma(-1) = ?$

$\varphi(t, -1) = (-1+1)e^{t} - 1 = -1$

$\gamma(-1) = \bigcup_{t \in \mathbb{R}} \varphi(t, -1) = \bigcup_{t \in \mathbb{R}} (-1) = \{-1\}$ here we didn't have to calculate γ^+, γ^- because $\varphi(t, -1) = -1$ constant and the answer would have been the same

$\gamma(0) = ?$

$\varphi(t, 0) = e^{t} - 1$

$\gamma^+(0) = \bigcup_{t \in [0, +\infty)} \varphi(t, 0) = [0, +\infty)$

$\gamma^-(0) = \bigcup_{t \in (-\infty, 0]} \varphi(t, 0) = (-1, 0]$

$\gamma(0) = \gamma^+(0) \cup \gamma^-(0) = (-1, 0] \cup [0, \infty) = (-1, \infty)$

$\gamma(-2) = ?$

$\varphi(t,-2) = -1 \cdot e^{t} - 1 = -e^{t} - 1$

$\gamma^{-}(-2) = \bigcup_{t\in(-\infty,0]} \varphi(t,-2) = [-2,-1)$

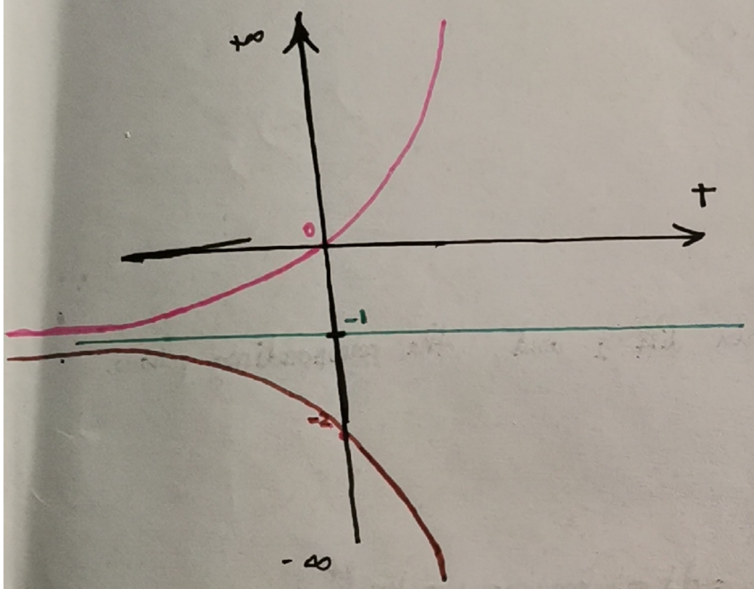
$\gamma^{+}(-2) = \bigcup_{t\in[0,\infty)} \varphi(t,-2) = (-\infty,-2]$

$\gamma_{(-2)} = \gamma^{-}(-2) \cup \gamma^{+}(-2) = (-\infty,-1)$

$\gamma(0) = (-1,+\infty)$ - $x=0$ $y \in (-1,\infty)$

$\gamma(-1) = \{-1\}$

$\gamma(-2) = (-\infty,-1)$



c) we have 3 cases

1. $m=-1$ $\varphi(t,-1) = -1$ $\gamma(-1) = \{-1\}$

2. $m>-1$, $\varphi(t,m) = \underbrace{(m+1)}_{>0}e^{t} - 1$

$\gamma^{+}(m) = \bigcup_{t\in[0,\infty)} \varphi(t,m) = [m,+\infty)$

$\gamma^{-}(m) = \bigcup_{t\in(-\infty,0]} \varphi(t,m) = (-1, m]$

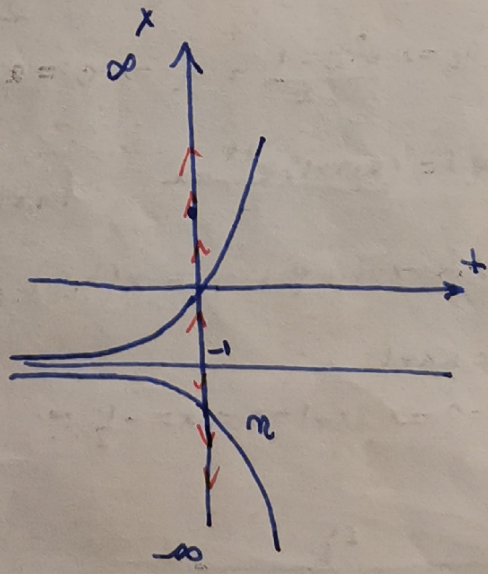
$\gamma(m) = \gamma^{+}(m) \cup \gamma^{-}(m) = (-1,\infty)$

3. $m<-1 \Rightarrow \varphi(t,m) = \underbrace{(m+1)}_{<0} e^{t} - 1$

$\gamma^{+}(m) = \bigcup_{t\in[0,\infty)} \varphi(t,m) = [-\infty,m]$

$\gamma^{-}(m) = \bigcup_{t\in(-\infty,0]} \varphi(t,m) = [m,-1)$

$\gamma(m) = \gamma^{+}(m) \cup \gamma^{-}(m) = (-\infty,-1)$



Seminar 12

Dynamical systems generated by autonomous scalar diff. eq.

$x' = f(x)$ $\quad$ $x = x(t)$. $\quad$, $\underline{f \in C^1}$

The flow = the saturated solution of the IVP:

$$\begin{cases} x' = f(x) \\ x(0) = \eta \ , \ \eta \in \mathbb{R}. \end{cases} \Rightarrow x(t, \eta)$$

$x(\cdot, \eta) : I_\eta \to \mathbb{R}$, I_η – maximal.

$I_\eta = (\alpha_\eta, \beta_\eta)$, $0 \in I_\eta$.

$W = \{ I_\eta \times \mathbb{R} \mid \eta \in \mathbb{R} \}$.

if $I_\eta = \mathbb{R} \to W = \mathbb{R}^2$

$\varphi : W \to \mathbb{R}$

$\varphi(t, \eta) = x(t, \eta)$

Properties of the flow:

1. $\varphi(0, \eta) = \eta$
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Orbits:

$\gamma^+(\eta) = \bigcup_{t \in [0, \beta_\eta)} \varphi(t, \eta)$ — the positive orbit of η

$\gamma^-(\eta) = \bigcup_{t \in (\alpha_\eta, 0]} \varphi(t, \eta)$ — the negative orbit of η.

$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta)$. the orbit of η.

Phase portrait (the phase line): the collection of all orbits with the describing direction.

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1) Let's consider the eq. $x' = x + 1$.

a) find the generated flow

b) find the orbits for $\eta = -1$, $\eta = 0$, $\eta = -2$.

c) find the phase portrait.

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a) $\begin{cases} x' = x+1 \\ x(0) = \eta . \end{cases}$

$\frac{dx}{dt} = x+1 \rightarrow \int \frac{dx}{x+1} = \int dt$

$\Rightarrow \ln|x+1| = t + \ln c .$

$x+1 = c \cdot e^t$

$\boxed{x(t) = c \cdot e^t - 1 , \ c \in \mathbb{R}}$ the gen. sol. of the diff. eq.

$x(0) = \eta \Rightarrow c - 1 = \eta \Rightarrow c = \eta + 1 .$

$\Rightarrow \ x(t,\eta) = (\eta+1) . e^t - 1 .$

$x(\cdot,\eta) : I_\eta \rightarrow \mathbb{R} \ , I_\eta$ maximal.

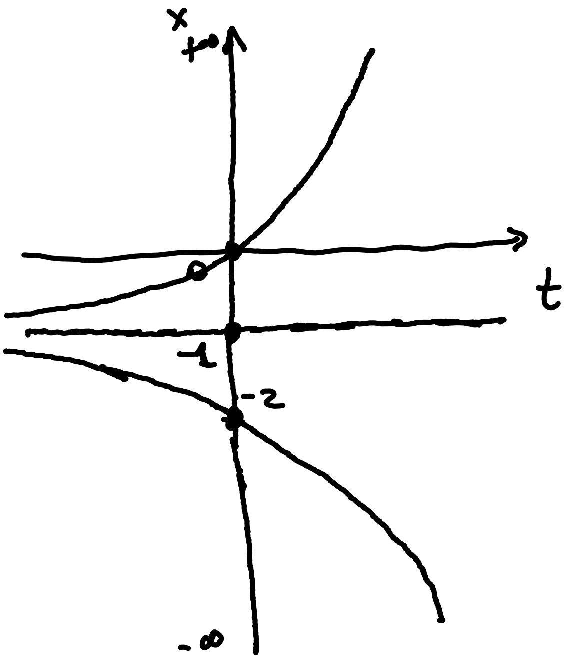
$\Rightarrow I_\eta = \mathbb{R} \ , \forall \eta \in \mathbb{R} .$

$\left.\begin{array}{l} \varphi(t,\eta) = x(t,\eta) = (\eta+1) e^t - 1 \\ \varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \end{array}\right\}$ the flow.

b) $\gamma(-1) = ?$ $\qquad \gamma(\eta) = \bigcup_{t \in I_\eta} \varphi(t,\eta)$

$\varphi(t,-1) = -1$

$$\gamma(-1) = \bigcup_{t\in\mathbb{R}} \varphi(t,-1) = \bigcup_{t\in\mathbb{R}} (-1) = \{-1\}.$$



$\gamma(0) = ?$

$\varphi(t,0) = e^t - 1$

$$\gamma^+(0) = \bigcup_{t\in[0,+\infty)} \varphi(t,0) = \overrightarrow{[0,+\infty)}$$

$$\gamma^-(0) = \bigcup_{t\in(-\infty,0]} \varphi(t,0) = \overrightarrow{(-1,0]}$$

$$\gamma(0) = \gamma^+(0) \cup \gamma^-(0) = \overrightarrow{(-1,+\infty)}$$

$\gamma(-2) = ?$

$$\varphi(t,-2) = -e^{t} - 1$$

$$\gamma^{+}(-2) = \bigcup_{t\in[0,+\infty)} \varphi(t,-2) = (-\infty,-2]$$

$$\gamma^{-}(-2) = \bigcup_{t\in(-\infty,0]} \varphi(t,-2) = [-2,-1)$$

$$\gamma(-2) = \gamma^{+}(-2) \cup \gamma^{-}(-2) = (-\infty,-1)$$

c) we have 3 cases:

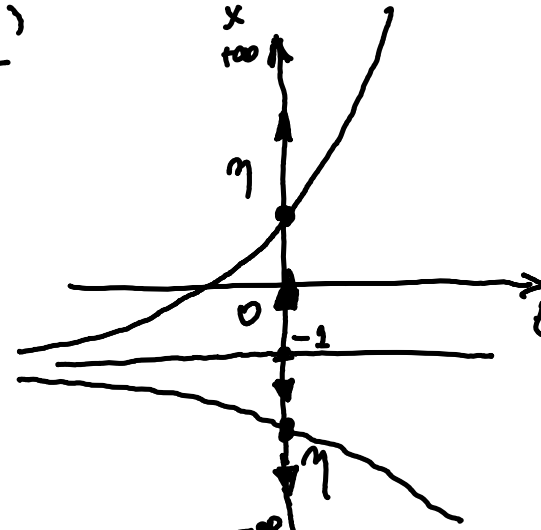
1. $\eta = -1 \Rightarrow \varphi(t,-1) \equiv -1.$

$$\gamma(-1) = \{-1\}$$

2. $\eta > -1$, $\varphi(t,\eta) = \underbrace{(\eta+1)}_{>0} e^{t} - 1$

$$\gamma^{+}(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = [\eta,+\infty)$$

$$\gamma^{-}(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = (-1,\eta]$$



$$\gamma(\eta) = \gamma^{+}(\eta) \cup \gamma^{-}(\eta) = (-1,+\infty)$$

3. $\eta < -1 \Rightarrow \varphi(t,\eta) = \underbrace{(\eta+1)}_{<0} e^{t} - 1$

$$\gamma^{+}(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = (-\infty, \eta] \quad \longleftarrow$$

$$\gamma^{-}(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = [\eta, -1) \quad \longleftarrow$$

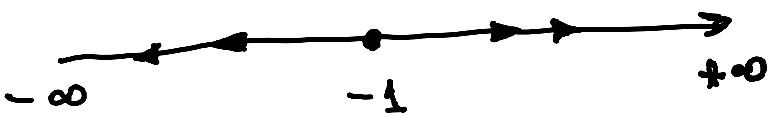
$$\gamma(\eta) = \gamma^{+}(\eta) \cup \gamma^{-}(\eta) = (-\infty, -1) \quad \longleftarrow$$

$f(x) = x+1$

$f(x) = 0 \Rightarrow x+1 = 0$

$x = -1$

x	$-\infty$		-1		$+\infty$
$f(x)$	$-$	$-$	0	$+$	$+$ $+$
	$\longleftarrow$				$\longrightarrow$



phase portrait.

$x' = f(x)$. $\quad f(x) = 0 \rightarrow x_1, x_2, \ldots, x_n$ real roots.

$x_1 < x_2 < \ldots < x_n$

x	$-\infty$	x_1		x_2		x_3	$\ldots$	x_n
$f(x)$	$-$	0	$+$	0	$+$	0	$\ldots$	0
	$\longleftarrow$		$\longrightarrow$		$\longleftarrow$			

2) Find the flow generated by the given diff. eq. and the corresponding phase portrait using the sign table of f.

a) $x' = 2x+1$

b) $x' = x^2$

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a) $\begin{cases} x' = 2x+1 \\ x(0) = \eta, \ \eta \in \mathbb{R}. \end{cases}$ $\qquad x(t) \equiv -\frac{1}{2}$

$\frac{dx}{dt} = 2x+1 \Rightarrow \qquad \frac{dx}{2x+1} = dt \,\Big|\cdot 2$

$\Rightarrow \int \frac{2\cdot dx}{2x+1} = \int 2\,dt \Rightarrow \ln(2x+1) = 2t + \ln c.$

$2x+1 = c\cdot e^{2t}$

$x(t) = \frac{c\cdot e^{2t} - 1}{2}, \ c \in \mathbb{R}.$

$x(0) = \eta \Rightarrow \frac{c-1}{2} = \eta \Rightarrow c - 1 = 2\eta$

$c = 2\eta + 1$

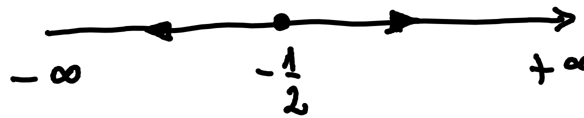
$\Rightarrow \boxed{x(t,\eta) = \frac{(2\eta+1)e^{2t} - 1}{2}} \qquad I_\eta = \mathbb{R}$

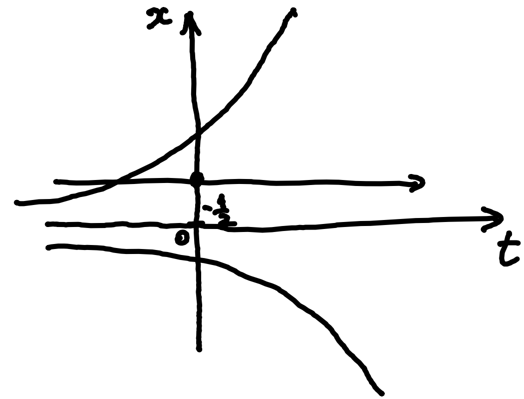
$\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \ \varphi(t,\eta) = x(t,\eta)$

$f(x)=2x+1$

$f(x)=0 \Rightarrow 2x+1=0 \Rightarrow x=-\frac{1}{2}$

x		$-\frac{1}{2}$	
$f(x)$	$- \ -$	0	$+ \ + \ +$
	$\longleftarrow$		$\longrightarrow$





b) $x' = x^2$

$$\begin{cases} x' = x^2 \\ x(0) = \eta \end{cases}$$

$x(t) \equiv 0$ is singular solution.

$$\frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt \quad \Big| \cdot (-1)$$

$$\int -\frac{dx}{x^2} = \int -dt \Rightarrow \frac{1}{x} = -t + c \Rightarrow x(t) = \frac{1}{-t+c}, \ c \in \mathbb{R}$$

gen. sol.

$$x(0)=\eta \Rightarrow \frac{1}{c}=\eta \Rightarrow c=\frac{1}{\eta}, \eta\neq 0.$$

$$\Rightarrow \begin{cases} x(t,\eta)=\dfrac{1}{-t+\frac{1}{\eta}}, & \eta\neq 0 \\ x(t,\eta)\equiv 0, & \eta=0 \end{cases}$$

$$\boxed{x(t,\eta)=\frac{\eta}{1-\eta t}, \quad \forall \eta\in\mathbb{R}}$$

$I_\eta = ?$ $\quad \eta=0 \Rightarrow x(t,0)\equiv 0 \Rightarrow \underline{I_0=\mathbb{R}}.$

$\eta\neq 0 \Rightarrow \left(-\infty,\frac{1}{\eta}\right)$ or $\left(\frac{1}{\eta},+\infty\right)$

$0\in I_\eta$

$$I_\eta=\begin{cases} \left(-\infty,\frac{1}{\eta}\right), & \eta>0 \\ \left(\frac{1}{\eta},+\infty\right), & \eta<0 \\ \mathbb{R}, & \eta=0 \end{cases}$$

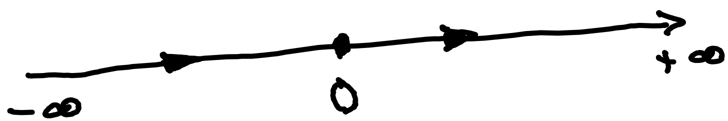
$\varphi: W\to\mathbb{R}$

$W=\{\underline{\underline{I_\eta}}\times\mathbb{R} \mid \eta\in\mathbb{R}\}$

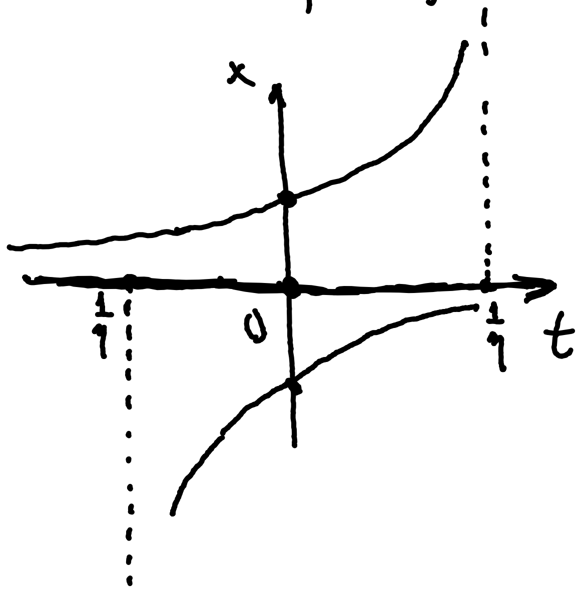
$$\varphi(t,\eta)=x(t,\eta)=\frac{\eta}{1-\eta t}$$

$f(x) = x^2$ $\quad$ $f(x) = 0 \Rightarrow x = 0$

x	$-\infty$		0		$+\infty$
$f(x)$		$+$	0	$+$	
		$\rightarrow$		$\rightarrow$	



phase portrait



$x' = f(x)$

sol. $x(t) \equiv x^*$ — equilibrium sol.

$x^* \in \mathbb{R}$ — the equilibrium point.

the equilibrium points are real solutions of the eq.

$$\boxed{f(x) = 0}$$

x^*

locally asymptotically stable

x^*

unstable eq. point

Theorem. (The Stability Theorem in the first approx.)

$f \in C^1$, x^* is an eq. point

a) If $f'(x^*) < 0 \Rightarrow x^*$ is locally as. stable

b) If $f'(x^*) > 0 \Rightarrow x^*$ is unstable.

3) Find the equilibrium points and study their stability for the eqs:

a) $x' = -2x$

b) $x' = 2+x$

c) $x' = x(1-x)$

d) $x' = \sin x$

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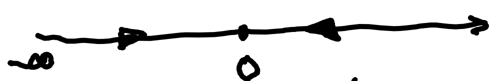
a) $x' = -2x$

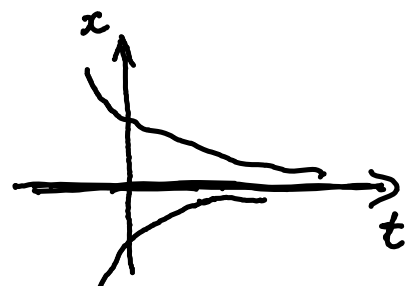
$f(x) = -2x \qquad f(x) = 0 \Rightarrow -2x = 0 \Rightarrow x^* = 0$ eq. point

$f'(x) = -2 \Rightarrow f'(0) = -2 < 0 \Rightarrow x^* = 0$ is asympt. stable

x	$-\infty$	0	$+\infty$
$f(x)$	$+\ +$	0	$-\ -\ -$
	$\longrightarrow$		$\longleftarrow$



asympt. stable



b) $x' = 2 + x$

$f(x) = 2 + x \Rightarrow f(x) = 0$

$2 + x = 0$

$x^* = -2$ eq. point.

$f'(x) = 1 > 0 \Rightarrow x^* = -2$ unstable.

x		-2	
$f(x)$	$- - -$	0	$+ + +$
	$\leftarrow$		$\rightarrow$

unstable

c) $x' = x(1-x)$

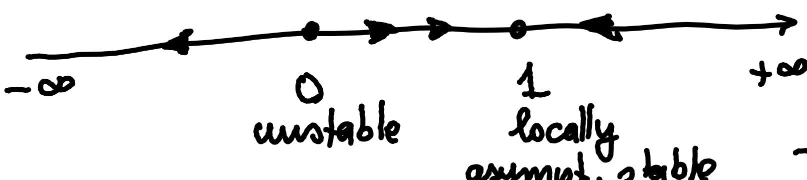
$f(x) = x(1-x) \qquad f(x) = 0 \Rightarrow x(1-x) = 0 \quad \begin{cases} x_1^* = 0 \\ x_2^* = 1 \end{cases}$ eq. points

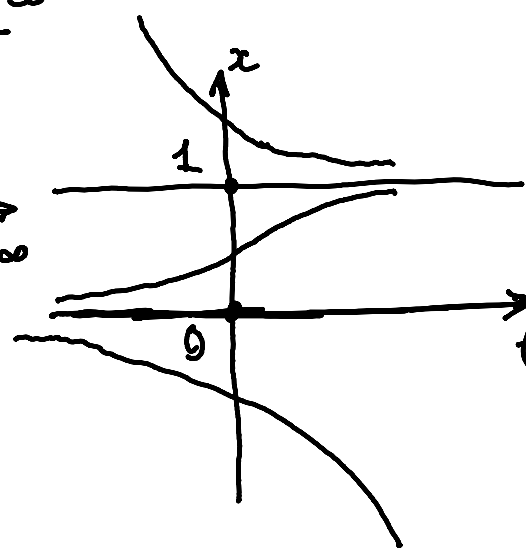
$f(x) = x - x^2 \rightarrow f'(x) = 1 - 2x$

$x_1^* = 0$: $f'(0) = 1 > 0 \Rightarrow x_1^* = 0$ is unstable

$x_2^* = 1$: $f'(1) = 1 - 2\cdot 1 = -1 < 0 \Rightarrow x_2^* = 1$ locally asympt. stable.

x	$-\infty$		0		1		$+\infty$
f(x)		$-\,-$	0	$+\,+$	0	$-\,-\,-$	
		←		→		←	





d) $x' = \sin x$.

$f(x) = \sin x$ $\qquad$ $f(x) = 0$

$\sin x = 0 \Rightarrow x_k^* = k\pi, \; k \in \mathbb{Z}$ eq. points.

$f'(x) = \cos x$

$$f'(x_k^*) = \cos(x_k^*) = \cos(k\pi) = \begin{cases} -1, & k \text{ odd.} \\ 1 & k \text{ even.} \end{cases}$$

x_k^* $\begin{cases} \text{locally asympt. stable for } k \text{ odd} \\ \text{unstable for } k \text{ even} \end{cases}$

x	-2π		$-\pi$		0		π		2π
$f(x)$	0	$+$	0	$-$	0	$+$	0	$-$	0
		$\rightarrow$		$\leftarrow$		$\rightarrow$		$\leftarrow$