SEMINAR 13

ECUATII

DYNAMICAL SYSTEMS GENERATED BY PLANAR SYSTEMS

$$(1)\ \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$$

FLOW = the saturated solution of the IVP

$$(2)\ \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \qquad \eta_1, \eta_2 \in \mathbb{R}$$

THEOREM if $f = (f_1, f_2) \in C^1$ then the IVP(2) has an unique saturated solution for every $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$

$x(t, \eta_1, \eta_2)$ $y(t, \eta_1, \eta_2) : I_\eta \to \mathbb{R}$

sat sol. of (2) $\Rightarrow I_\eta$ maximal.

$$\left.\begin{matrix} I_\eta = (\alpha_\eta, \beta_\eta) \\ 0 \in I_\eta \end{matrix}\right\} \Rightarrow \alpha_\eta < 0 < \beta_\eta$$

$W = \{ I_\eta \times \mathbb{R}^2 \mid \eta \in \mathbb{R}^2 \}$

$\varphi : W \to \mathbb{R}^2$

$\varphi(t, \eta_1, \eta_2) = (x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2))$

if $I_\eta = \mathbb{R}$, $\forall\, \eta \in \mathbb{R}^2 \Rightarrow W = \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$

φ - the flow generated by (1)

Properties:

1. $\varphi(0, \eta) = \varphi(0, \eta_1, \eta_2) = (\eta_1, \eta_2)$
2. $\varphi(t+s, \eta) = \varphi(t, \varphi(s, \eta))$
3. φ is continuous

ORBITS $\gamma^+(\eta_1,\eta_2) = \bigcup\limits_{t\in[0,\beta_\eta)} \varphi(t,\eta)$ positive orbit of $\eta=(\eta_1,\eta_2)$

$\gamma^-(\eta_1,\eta_2) = \bigcup\limits_{t\in(\alpha_\eta,0]} \varphi(t,\eta)$ negative orbit

$\gamma(\eta) = \gamma(\eta_1,\eta_2) = \gamma^+(\eta) \cup \gamma^-(\eta)$

$\longrightarrow$ the orbit of $\eta=(\eta_1,\eta_2)$

PHASE PORTRAIT = collection of all orbits with their describing sense

1. Let's consider the system:

$$\begin{cases} x' = -x \\ y' = -2y \end{cases}$$

a) find the flow generated

b) find the orbits of $(0,0), (-1,0), (0,1), (1,1)$

c) find the phase portrait

a) $$\begin{cases} x' = -x \\ y' = -2y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \qquad \eta_1,\eta_2 \in \mathbb{R}$$

$x' = -x \Rightarrow \frac{dx}{dt} = -x \Rightarrow \frac{dx}{x} = -1 \Rightarrow \ln x = -t \Rightarrow x = e^{-t}\cdot c$

$y' = -2y \Rightarrow \dots\dots \Rightarrow y = e^{-2t}\cdot c$

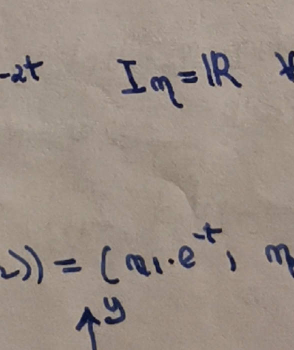
$x(0) = \eta_1 \Rightarrow c = \eta_1 \Rightarrow x(t,\eta_1,\eta_2) = \eta_1 \cdot e^{-t}$

$y(0) = \eta_2 \Rightarrow c = \eta_2 \Rightarrow y(t,\eta_1,\eta_2) = \eta_2 \cdot e^{-2t}$ $\qquad I_\eta = \mathbb{R} \;\; \forall \eta = (\eta_1,\eta_2) \in \mathbb{R}^2$

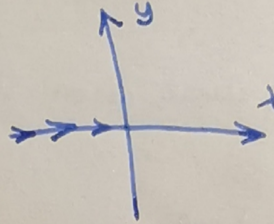
$\Rightarrow \varphi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$

$\varphi(t,\eta_1,\eta_2) = (x(t,\eta_1,\eta_2), y(t,\eta_1,\eta_2)) = (\eta_1\cdot e^{-t}, \eta_2\cdot e^{-2t})$

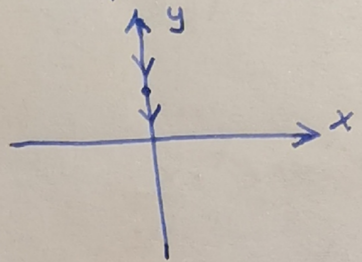
b) $\gamma(0,0) = \bigcup\limits_{t\in\mathbb{R}} \varphi(t,0,0) = \bigcup\limits_{t\in\mathbb{R}} (0,0) = \{(0,0)\}$



$\gamma(-1,0) = \bigcup_{t\in\mathbb{R}} \varphi(t,-1,0) = \bigcup_{t\in\mathbb{R}} (-e^{-t},0) = \{(x,0) \mid x<0\}$



$\gamma(0,1) = \bigcup_{t\in\mathbb{R}} \varphi(t,0,1) = \bigcup_{t\in\mathbb{R}} (0, e^{-2t}) = \{(0,y) \mid y>0\}$



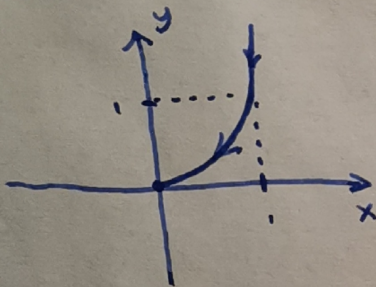
$\gamma(1,1) = \bigcup_{t\in\mathbb{R}} \varphi(t,1,1) = \bigcup_{t\in\mathbb{R}} (e^{-t}, e^{-2t}) = \{(x,y) \mid x>0, y>0\}$

$M\in\gamma(1,1)$ $\begin{cases} x_M = e^{-t} \\ y_M = e^{-2t} \end{cases}$ $t\in\mathbb{R}$

$y_M = e^{-2t} = (e^{-t})^2 = x_M^2$

the orbit $\gamma(1,1)$ has the equation $y = x^2$

with $x>0$, $y>0$



=> phase portrait

1. $\eta_1 = \eta_2 = 0$ $\gamma(0,0) = \{(0,0)\}$

2. $\eta_1 = 0$, $\eta_2 \neq 0$

$\gamma(0,\eta_2) = \bigcup_{t\in\mathbb{R}} \varphi(t,0,\eta_2) = \bigcup_{t\in\mathbb{R}} (0, \eta_2 e^{-2t}) = \left\{(0,y) \;\middle|\; \begin{matrix} y>0 \text{ if } \eta_2>0 \\ y<0 \text{ if } \eta_2<0 \end{matrix}\right\}$

3. $\eta_1 \neq 0$ $\eta_2 \neq 0$

$\gamma(\eta_1,0) = \bigcup_{t\in\mathbb{R}} \varphi(t,\eta_1,0) = \bigcup_{t\in\mathbb{R}} (\eta_1 e^{-t}, 0) = \left\{(x,0) \;\middle|\; \begin{matrix} x>0, & \eta_1>0 \\ x<0 & \eta_1<0 \end{matrix}\right\}$

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$\gamma(m_1, m_2)$ is a curve given by the parametric eqs.

$$\begin{cases} x = m_1 e^{-t} \\ y = m_2 e^{-2t} \end{cases} \quad t \in \mathbb{R}$$

$$x = m_1 e^{-t} \Rightarrow e^{-t} = \frac{x}{m_1}$$

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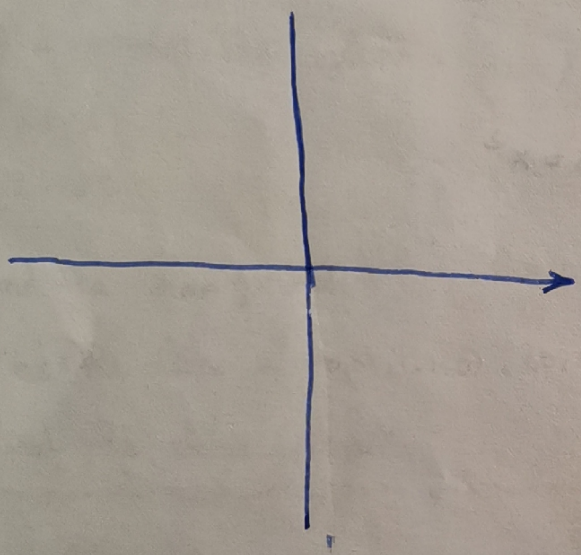
$\Rightarrow \gamma(m_1, m_2)$ is given by the equation

$$y = \frac{m_2}{m_1^2} \cdot x^2 \quad - \text{ a parabola}$$



the phase portrait

Seminar 13

Dynamical systems generated by planar systems

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Theorem: if. $f = (f_1, f_2) \in C^1$ then the IVP(2) has an unique saturated solution for every $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$

$$x(t,\eta_1,\eta_2),\ y(t,\eta_1,\eta_2) : I_\eta \to \mathbb{R}$$

sat. sol. of (2) $\Longrightarrow$ I_η - maximal.

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<u>Orbits</u>: $\gamma^+(\eta_1,\eta_2) = \bigcup_{t\in[0,\beta_\eta)} \varphi(t,\eta)$ positive orbit of $\eta=(\eta_1,\eta_2)$

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a) $\begin{cases} x' = -x \\ y' = -2y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$

$x' = -x$

$x(t) = c_1 \cdot e^{-t}$

$x(0) = \eta_1$

$\Downarrow$

$c_1 = \eta_1$

$y' = -2y$

$y(t) = c_2 \cdot e^{-2t}$

$y(0) = \eta_2$

$\Downarrow$

$c_2 = \eta_2$

$\Rightarrow \begin{cases} x(t, \eta_1, \eta_2) = \eta_1 e^{-t} \\ y(t, \eta_1, \eta_2) = \eta_2 e^{-2t} \end{cases}$ $\quad I_\eta = \mathbb{R}, \ \forall \eta = (\eta_1, \eta_2) \in \mathbb{R}^2$

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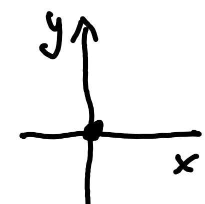
$$\varphi(t, \eta_1, \eta_2) = \big(x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2) \big) =$$

$$= \left(\eta_1 e^{-t}, \eta_2 e^{-2t} \right).$$

b) $\gamma(0,0) = ?$

$$\gamma(0,0) = \bigcup_{t \in \mathbb{R}} \varphi(t, 0, 0) = \bigcup_{t \in \mathbb{R}} (0,0) = \{(0,0)\}$$



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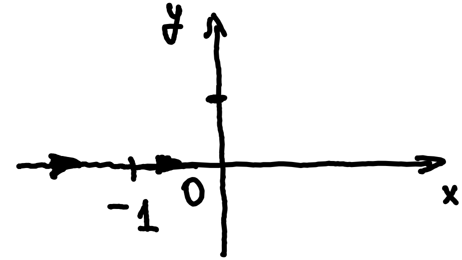

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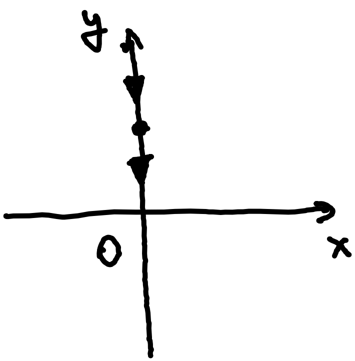

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$M \in \gamma(1,1)$ $\quad \begin{cases} x_M = e^{-t} \\ y_M = e^{-2t} \end{cases}, t \in \mathbb{R}$

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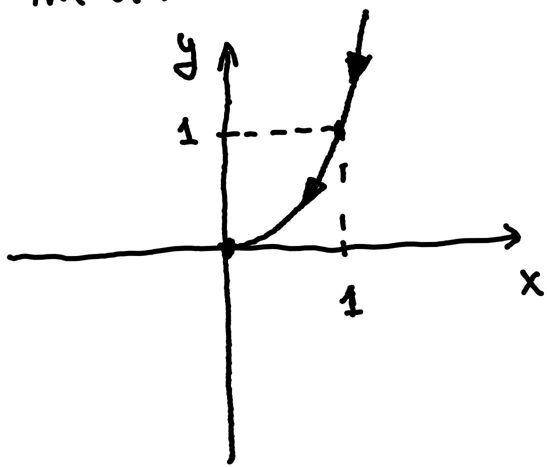
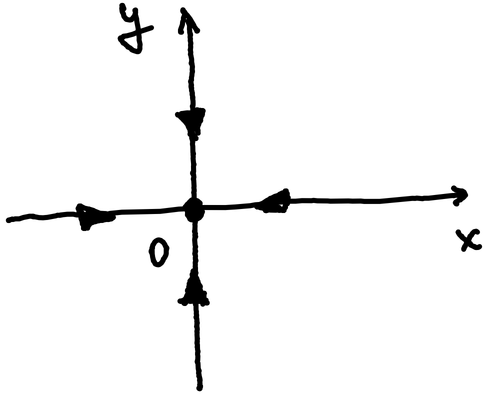


c) phase portrait.



1. $\eta_1 = \eta_2 = 0$: $\gamma(0,0) = \{(0,0)\}$

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$\gamma(\eta_1, \eta_2)$ is a curve given by the parametric eqs.

$$\begin{cases} x = \eta_1 e^{-t} \\ y = \eta_2 e^{-2t} \end{cases}, t \in \mathbb{R}$$

$$x = \eta_1 e^{-t} \Rightarrow e^{-t} = \frac{x}{\eta_1}$$

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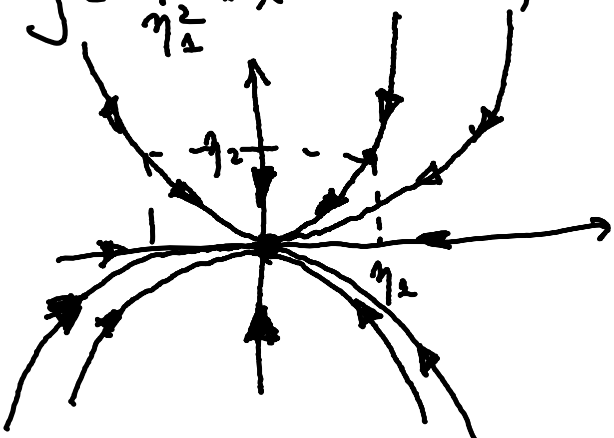


the phase portrait.

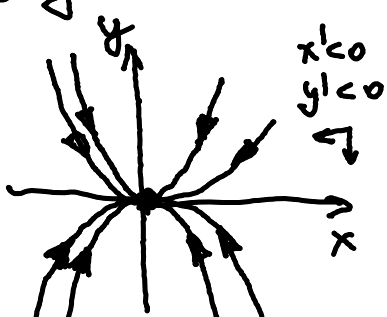
$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \qquad \left.\begin{array}{l} \dfrac{dx}{dt} = f_1(x,y) \\[2ex] \dfrac{dy}{dt} = f_2(x,y) \end{array}\right\} \Rightarrow \boxed{\underbrace{\frac{dx}{dy}}_{x'(y)} = \frac{f_1(x,y)}{f_2(x,y)}}$$

$$\boxed{\underbrace{\frac{dy}{dx}}_{y'(x)} = \frac{f_2(x,y)}{f_1(x,y)}}$$ the differential eq. of the orbits.

$$\begin{cases} x' = -x \\ y' = -2y \end{cases} \Rightarrow \frac{dx}{dy} = \frac{-x}{-2y} \Rightarrow \frac{dx}{dy} = \frac{x}{2y} \Rightarrow$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2\,dx}{x} \Rightarrow \ln y = 2\ln x + \ln c$$

$$\boxed{y = cx^2, \; c \in \mathbb{R}}$$ parabolas.



2) Find the phase portrait of the following systems using the diff. eq. of the orbits.

a) $\begin{cases} x' = x \\ y' = -2y \end{cases}$

b) $\begin{cases} x' = y \\ y' = -a^2 \cdot x \end{cases}$, $a \in \mathbb{R}^*$

c) $\begin{cases} x' = x \\ y' = x + 2y \end{cases}$

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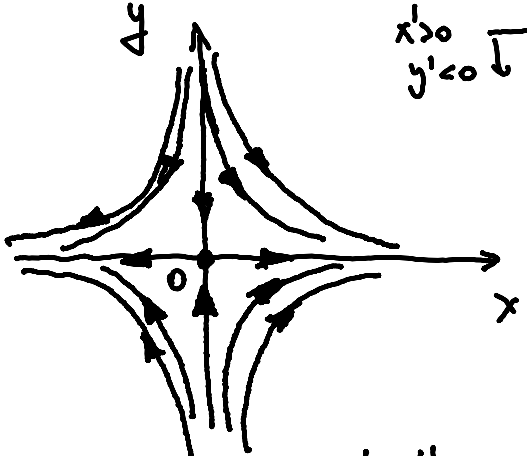
a) $\begin{cases} x' = x \\ y' = -2y \end{cases}$ $\quad \dfrac{dx}{dy} = \dfrac{x}{-2y}$ the diff. eq. of the orbits.

$$\int \frac{dy}{y} = \int -\frac{2\,dx}{x}$$

$$\ln y = -2 \ln x + \ln c$$

$$\boxed{y = c \cdot x^{-2}, \ c \in \mathbb{R}}$$



phase portrait

$x' > 0 \rightarrow$

$y' < 0 \downarrow$

$y = \frac{-c}{x^2}$

b) $\begin{cases} x' = y \\ y' = -a^2 x \end{cases}$

$$\frac{dx}{dy} = \frac{y}{-a^2 x}$$

$$y\,dy = -a^2 x \cdot dx \quad | \cdot 2$$

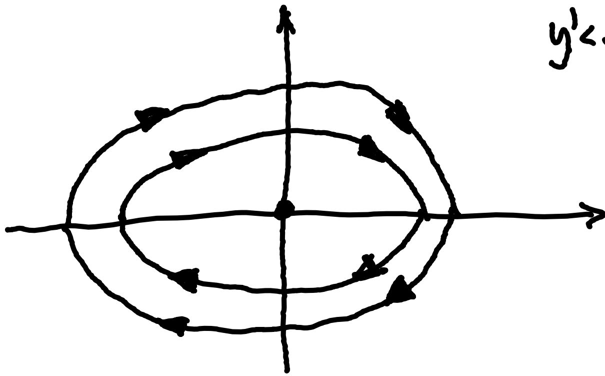
$$\int 2y\,dy = \int -2a^2 x \cdot dx$$

$$y^2 = -a^2 x^2 + c$$

$$\boxed{a^2 x^2 + y^2 = -c, \quad c \in \mathbb{R}}$$

$$\Rightarrow \left(\frac{x}{\frac{\sqrt{-c}}{a}}\right)^2 + \left(\frac{y}{\sqrt{-c}}\right)^2 = 1 \quad \text{ellipses.}$$

$x' > 0 \quad \rightarrow$
$y' < 0 \quad \downarrow$



c) $\begin{cases} x' = x \\ y' = x + 2y \end{cases}$

$$\frac{dx}{dy} = \frac{x}{x+2y}$$

$$\frac{dy}{dx} = \frac{x+2y}{x}$$

$$\underbrace{\frac{dy}{dx}}_{y'(x)} = 1 + \frac{2}{x}\cdot y$$

$y' = 1 + \frac{2}{x} \cdot y \quad \Rightarrow \quad \boxed{y' - \frac{2}{x} y = 1}$ the diff. eq. of the orbits.

$y' - \frac{2}{x} y = 0$

$y' = \frac{2}{x} y$

$\frac{dy}{dx} = \frac{2}{x} \cdot y \quad \Rightarrow \int \frac{dy}{y} = \int \frac{2}{x} \cdot dx$

$\ln y = 2 \ln x + \ln c$

$y_0(x) = c \cdot x^2, \, c \in \mathbb{R}.$

$y_p(x) = c(x) \cdot x^2$

$y_p' - \frac{2}{x} \cdot y_p = 1$

$c'(x) \cdot x^2 + \cancel{c(x) \cdot 2x} - \cancel{\frac{2}{x} \cdot c(x)} \cdot x^2 = 1$

$c'(x) \cdot x^2 = 1 \quad \Rightarrow \quad c'(x) = \frac{1}{x^2} \quad \Rightarrow \quad c(x) = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$\Rightarrow \quad y_p(x) = c(x) \cdot x^2 = -\frac{1}{x} \cdot x^2 = -x.$

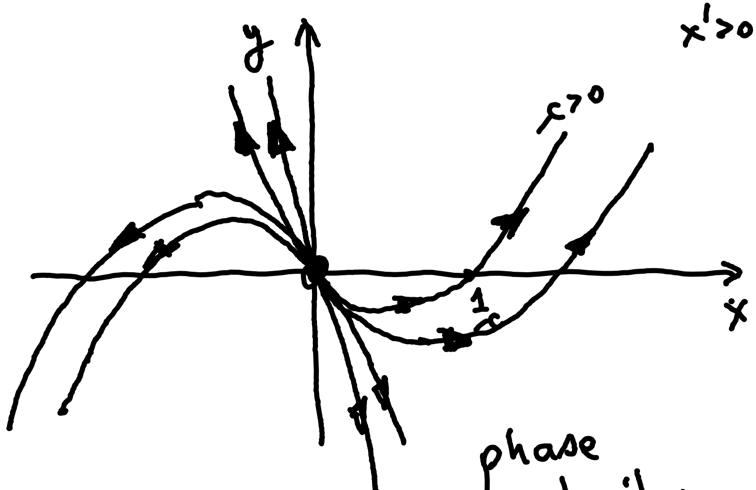
$y = y_0 + y_p$

$y(x) = c \cdot x^2 - x \, , \; c \in \mathbb{R} \, .$

$y = cx^2 - x$

$y = x(cx - 1)$

$x = 0 \, , \; x = \frac{1}{c} \, .$



$x' > 0 \longrightarrow \begin{cases} x' = x \\ y' = x + 2y \end{cases}$

phase portrait.