Seminar 14

Dynamical systems generated by the planar systems. Equilibrium points. Stability of equilibrium points

$$(1)\ \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \qquad x = x(t),\ y = y(t)$$

Equilibrium solution = constant solution

$$\begin{cases} x(t) \equiv x^* \\ y(t) \equiv y^* \end{cases}$$

the point $X^*(x^*, y^*)$ = equilibrium point.

the equilibrium point $X^*(x^*, y^*)$ is a solution of the system: $\underline{x^*, y^* \in \mathbb{R}}$.

$$(2)\ \begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$$

(x^*, y^*) is a real sol. of the system (2)

Stability.

I Linear case.

$$(3) \begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$(3) \Leftrightarrow (3')$ $\quad X' = AX.$

$X^*(0,0)$ – is an equilibrium point of (3).

Theorem (The Stability Theorem in the linear case)

Let's consider the system (3). Then:

a) If $Re\,\lambda < 0$, $\forall \lambda$ eigenvalue of $A \Rightarrow (0,0)$ is asymptotically stable

b) If $Re\,\lambda \le 0$, $\forall \lambda$ eigenvalue of A, but the equality with 0 holds for simple eigenvalue $\Rightarrow (0,0)$ is locally stable

c) If $\exists \lambda$, with $Re\,\lambda > 0$ or $\exists \lambda$ with $Re\,\lambda = 0$ and λ is simple eigenvalue $\Rightarrow (0,0)$ is unstable.

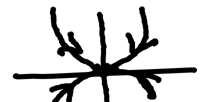
$\boxed{\det(\lambda I_2 - A) = 0}$ the characteristic eq.

The classification of the eq. point $(0,0)$.

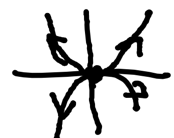
the point $(0,0)$ is.

- node if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \cdot \lambda_2 > 0$

  if $\lambda_1, \lambda_2 < 0 \Rightarrow$ sink node (as. stable)

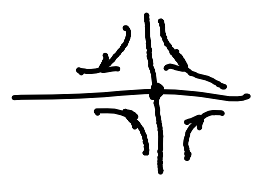
  

  if $\lambda_1, \lambda_2 > 0 \Rightarrow$ source node (unstable node)

- saddle if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \cdot \lambda_2 < 0$

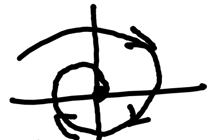
  always the saddle point is unstable

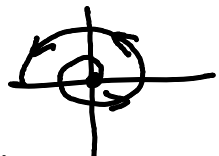
- spiral (focus) $\lambda_{1,2} \in \mathbb{C}$

$\lambda_{1,2} = \alpha \pm i\beta$. and $\alpha \neq 0$.

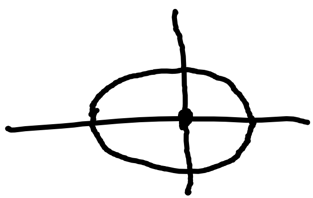
- if $\alpha < 0 \Rightarrow$ spiral sink $\Rightarrow$ as. stable spiral.



- if $\alpha > 0 \Rightarrow$ spiral source $\Rightarrow$ unstable



- center $\lambda_{1,2} \in \mathbb{C}$ $\lambda_{1,2} = \pm i\beta$ $(\alpha = 0)$



locally stable

II Non linear case.

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \qquad X^*(x^*, y^*)$$

$(x^*, y^*) \in \mathbb{R}^2$ is a sol. of the system

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$$

$f = (f_1, f_2) \quad \Rightarrow \quad X' = f(X)$. $\simeq$ $Y' = J_f(X^*) . Y$.

the linearized system in X^*.

$X = \begin{pmatrix} x \\ y \end{pmatrix}$

where $J_f(X) = J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$.

<u>Theorem</u> (The Stability Theorem in the first approximation)
Let's consider the nonlinear syst. (1) and $X^*(x^*,y^*)$ an eq. point of (1). Then:

a) If $\operatorname{Re}\lambda < 0$, $\forall\lambda$ eigenvalue of $J_f(X^*) = J_f(x^*,y^*)$
$\Rightarrow X^*(x^*,y^*)$ is locally asimptotically stable

b) If $\exists\lambda$ with $\operatorname{Re}\lambda > 0$ an eigenvalue of $J_f(X^*)$
$\Rightarrow X^*(x^*,y^*)$ is unstable.

<u>Exercise</u>: Find the equilibrium points and study their stability for:

a) $\begin{cases} x' = x \\ y' = -2y \end{cases}$

b) $\begin{cases} x' = y \\ y' = -a^2 \cdot x \end{cases}$, $a \in \mathbb{R}^*$

c) $\begin{cases} x' = x + 5y \\ y' = 5x + y \end{cases}$

d) $\begin{cases} x' = 1 - xy \\ y' = x - y^3 \end{cases}$

e) $\begin{cases} x' = y \\ y' = 2x^3 + x^2 - x \end{cases}$

a) $\begin{cases} x' = x \\ y' = -2y \end{cases}$ linear system $\Rightarrow$ $(0,0)$ is equil. point.

$x' = f_1(x,y)$
$y' = f_2(x,y)$

$\dfrac{dx}{dy} = \dfrac{f_1}{f_2}$ the diff. eq. of the orbits.

$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

$\det(\lambda I_2 - A) = 0$

$\begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 2 \end{vmatrix} = 0$

$(\lambda - 1)(\lambda + 2) = 0$

$\Rightarrow \lambda_1 = 1$

$\lambda_2 = -2$

$\lambda_1 > 0 \Rightarrow$

$\Rightarrow (0,0)$ is unstable of saddle type.

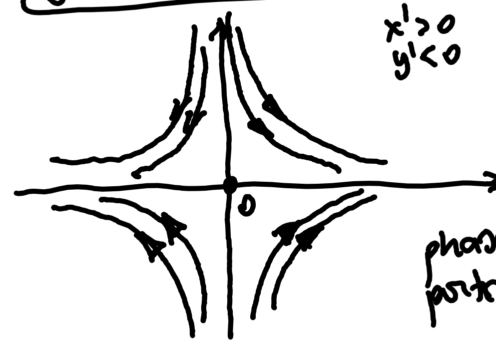
$f_1(x,y) = x$
$f_2(x,y) = -2y$

$\dfrac{dx}{dy} = \dfrac{x}{-2y} \Rightarrow \displaystyle\int \frac{dy}{y} = \int \frac{-2dx}{x}$

$\ln y = -2 \ln x + \ln c$

$\boxed{y = c \cdot x^{-2},\ c \in \mathbb{R}}$

$x' > 0$
$y' < 0$



phase portrait.

b) $\begin{cases} x' = y \\ y' = -a^2 x \end{cases}$ $a \in \mathbb{R}^*$

linear system.

$$A = \begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix}$$

$$\det(\lambda I_2 - A) = 0$$

$$\begin{vmatrix} \lambda & -1 \\ a^2 & \lambda \end{vmatrix} = 0$$

$\Rightarrow$ $\lambda^2 + a^2 = 0$

$$\lambda^2 = -a^2$$

$$\lambda_{1,2} = \pm i a$$

$\left.\begin{array}{l} \mathrm{Re}\,\lambda_{1,2} = 0 \\ \lambda_1, \lambda_2 \text{ are simple} \end{array}\right\} \Rightarrow$

$\Rightarrow$ $(0,0)$ is locally stable of center type

$$\frac{dx}{dy} = \frac{y}{-a^2 x}$$

$$y\,dy = -a^2 x\,dx \quad | \cdot 2$$

$$\int 2y\,dy = \int -2a^2 x\,dx$$

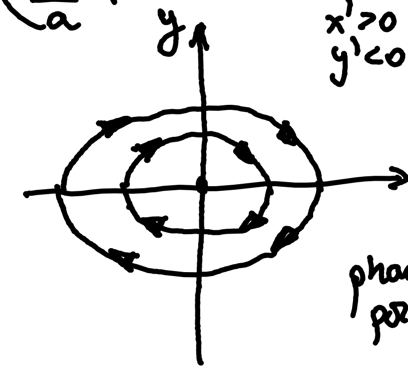
$$y^2 = -a^2 x^2 + c.$$

$$\boxed{a^2 x^2 + y^2 = c}, \quad c \in \mathbb{R}$$

the equation of the orbits.

$$\left(\frac{x}{\frac{\sqrt{c}}{a}}\right)^2 + \left(\frac{y}{\sqrt{c}}\right)^2 = 1 \quad \text{ellipses.}$$

$x' > 0$
$y' < 0$



phase portrait.

c) $\begin{cases} x' = x + 5y \\ y' = 5x + y \end{cases}$

$$A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$$

$$\det(\lambda I_2 - A) = 0$$

$$\begin{vmatrix} \lambda - 1 & -5 \\ -5 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 1)^2 - 25 = 0$$

$$(\lambda - 1)^2 = 25$$

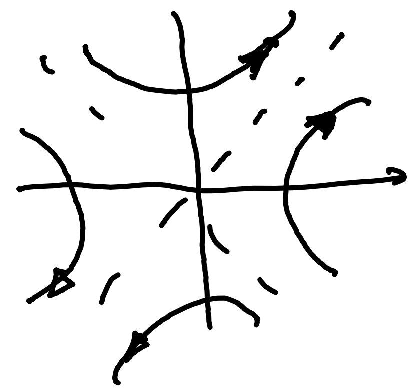
$$\lambda - 1 = \pm 5$$

$$\lambda = 1 \pm 5$$

$$\lambda_1 = 6$$

$$\lambda_2 = -4$$

$\lambda_1 = 6 > 0 \Rightarrow (0,0)$ is unstable of saddle type



d) $\begin{cases} x' = 1 - xy \\ y' = x - y^3 \end{cases}$ nonlinear system

$f_1(x,y) = 1 - xy$

$f_2(x,y) = x - y^3$

Eq. points:

$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 1 - xy = 0 \\ x - y^3 = 0 \end{cases} \Rightarrow \boxed{x = y^3}$

$\Rightarrow 1 - y^3 \cdot y = 0 \Rightarrow 1 - y^4 = 0 \Rightarrow y^4 = 1$

$y_{1,2} = \pm 1$

$y_{3,4} = \pm i \notin \mathbb{R}.$

$y_1 = 1 \Rightarrow x = y^3 = 1 \Rightarrow X_1^*(1,1)$

$y_2 = -1 \Rightarrow x = y^3 = -1 \Rightarrow X_2^*(-1,-1)$

Stability:

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}$$

$X_1^*(1,1)$: $J_f(1,1) = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$

$$\det\left(\lambda I_2 - J_f(1,1)\right) = 0$$

$$\begin{vmatrix} \lambda+1 & 1 \\ -1 & \lambda+3 \end{vmatrix} = 0 \Rightarrow (\lambda+1)(\lambda+3)+1 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + \lambda + 3 + 1 = 0 \Rightarrow \lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda+2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -2$$

$\mathrm{Re}\,\lambda_{1,2} < 0 \Rightarrow X_1^*(1,1)$ is locally asympt. stable of node type (sink node)

$\underline{X_2^*(-1,-1)}$: $\quad J_f(-1,-1) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$

$$\det\left(\lambda I_2 - J_f(-1,-1)\right) = 0$$

$$\begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda + 3 \end{vmatrix} = 0 \quad \to (\lambda - 1)(\lambda + 3) - 1 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda - \lambda - 3 - 1 = 0$$

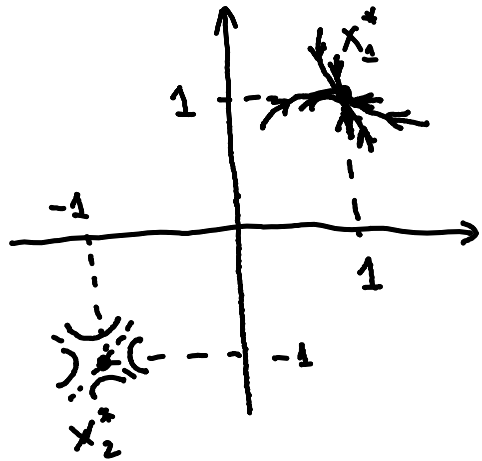
$$\lambda^2 + 2\lambda - 4 = 0$$

$$\Delta = 4 - 4.(-4) = 4 + 16 = 20.$$

$$\lambda_{1,2} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

$$\left.\begin{array}{l} \lambda_1 = -1 + \sqrt{5} > 0 \\ \lambda_2 < 0 \end{array}\right\}$$

$Re\,\lambda_1 > 0 \Rightarrow X_2^*(-1,-1)$ is unstable of saddle type.



e) $\begin{cases} x' = y \\ y' = 2x^3 + x^2 - x \end{cases}$ nonlinear system

$f_1(x, y) = y$

$f_2(x, y) = 2x^3 + x^2 - x$

Eq. points : $\begin{cases} f_1 = 0 \\ f_2 = 0 \end{cases} \to \begin{cases} \boxed{y = 0} \\ 2x^3 + x^2 - x = 0 \end{cases}$

$2x^3+x^2-x=0$

$x(2x^2+x-1)=0$

$x_1=0$ $\qquad$ $2x^2+x-1=0$

$\Delta = 1-4\cdot 2(-1) = 9$

$x_{2,3} = \dfrac{-1\pm 3}{4}$ $\quad x_2=\frac{1}{2}$, $\quad x_3=-1$

$\Rightarrow X_1^*(0,0)$, $X_2^*(\frac{1}{2},0)$, $X_3^*(-1,0)$ eq. points.

Stability

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6x^2+2x-1 & 0 \end{pmatrix}$$

$X_1^*(0,0)$: $\quad J_f(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\det(\lambda I_2 - J_f(0,0)) = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0$

$\lambda^2+1=0 \Rightarrow \lambda_{1,2} = \pm i$

$Re\,\lambda_{1,2}=0$, the system is nonlinear $\Rightarrow$

$\Rightarrow$ we cannot apply the Stab. Th. in the first approx.

$\Rightarrow$ we cannot say anything about the stab. of $X_1^*(0,0)$

$\underline{X_2^*(\frac{1}{2},0)}$: $J_f(\frac{1}{2},0) = \begin{pmatrix} 0 & 1 \\ \frac{3}{2}+1-1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{3}{2} & 0 \end{pmatrix}$

$\Rightarrow \det(\lambda I_2 - J_f(\frac{1}{2},0)) = 0$

$$\begin{vmatrix} \lambda & -1 \\ -\frac{3}{2} & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{3}{2} = 0$$

$$\lambda^2 = \frac{3}{2}$$

$$\lambda_{1,2} = \pm\sqrt{\frac{3}{2}}$$

$\underline{\lambda_1} > 0 \Rightarrow X_2^*$ is unstable of saddle type.

...

$\underline{X_3^*(-1,0)}$: homework.