

SOLVABLE FIRST ORDER DIFFERENTIAL EQUATIONS1. LINEAR DIFFERENTIABLE EQUATIONS

General form $y' + P(x)y = Q(x)$ P, Q are continuous functions

$$L: C^1[a, b] \rightarrow C[a, b]$$

$$y \rightarrow y' + P(x)y$$

linear operator

$Ly = Q$ nonhomogeneous equation

$Ly = 0$ homogeneous equation

general solution: $y = y_0 + y_p$ where:

y_0 - the general solution of the homogeneous equation
 $y' + P \cdot y = 0$

y_p - a particular solution of the nonhomogeneous eq
 $y' + P \cdot y = Q$

• can be found using the variation of the constant method (Lagrange method)

Exercise 1:

a) $y' + y \cdot \tan(x) = \frac{1}{\cos(x)}$

b) $y' + \frac{2}{x} \cdot y = x^3$

c) $y' + 2x \cdot y = 2xe^{-x^2}$

d) $xy' - y + x = 0$

e) $y' - y = \sin x$

f) $y' + \frac{x}{1-x^2} y = x + \arcsin x$

$$a) y' + y \cdot \tan(x) = \frac{1}{\cos(x)}$$

$$y = y_0 + y_p$$

$y' + y \cdot \tan x = 0$ the homogeneous eq

$$y' = \underbrace{-y}_{g(y)} \cdot \underbrace{\tan x}_{f(x)}$$

$$y' = \frac{dy}{dx} = -y \cdot \tan x \Rightarrow \frac{dy}{y} = -dx \tan x \Rightarrow \int \frac{dy}{y} = - \int \tan x \, dx$$

$$\Rightarrow \ln|y| = -(-\ln|\cos x| + \ln c) \Rightarrow$$

$$\ln|y| = \ln|\cos x| + \ln c \Rightarrow$$

$$\ln|y| = \ln|\cos x \cdot c| \Rightarrow$$

$$\boxed{y_0 = \cos x \cdot c}, c \in \mathbb{R}$$

general solution of the homogeneous equation

$y_p = ?$

$$y' + y \cdot \tan(x) = \frac{1}{\cos x}$$

$$y_p \in x) = c(x) \cdot \cos x$$

$$y'_p = c'(x) \cdot \cos x - \sin x \cdot c(x)$$

$$y'_p + y_p \cdot \tan(x) = \frac{1}{\cos x}$$

$$\cancel{c'(x) \cdot \cos x} - \cancel{\sin x \cdot c(x)} + \cancel{c(x) \cdot \cos x} - \cancel{\sin x \cdot c(x)} = \frac{1}{\cos x}$$

$$c'(x) \cdot \cos x = \frac{1}{\cos x}$$

$$c'(x) = \frac{1}{\cos^2 x} \Rightarrow c(x) = \int \frac{1}{\cos^2 x} \, dx = \tan x$$

$$y_p = c(x) \cdot \cos x = \tan x \cdot \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x$$

general solution: $y = y_p + y_0$

$$y(x) = \cos x \cdot c(x) + \sin x$$

$$b) \underbrace{y' + \frac{2}{x} \cdot y}_{P(x)} = \underbrace{x^3}_{Q(x)}$$

nonhomogeneous diff eq.

I $y' + \frac{2}{x} \cdot y = 0$ • homogeneous eq

$$y' = -\frac{2}{x} \cdot y$$

$\underbrace{y}_{f(x)} \quad \underbrace{g(x)}$

$$y' = \frac{dy}{dx} = -\frac{2}{x} \cdot y \Rightarrow \frac{dy}{y} = -\frac{2}{x} dx \Rightarrow \int \frac{dy}{y} = \int -\frac{2}{x} dx \Rightarrow \ln|y| = \ln|x| \cdot -2 \Rightarrow \ln c$$

$$\Rightarrow \ln|y| = \ln(x^{-2} \cdot c) \Rightarrow y = x^{-2} \cdot c$$

$$y_0(x) = x^{-2} \cdot c, \quad c \in \mathbb{R}$$

II $y_p = ? \quad y_p(x) = \frac{c(x)}{x^2} \quad y'_p = \frac{c'(x) \cdot x^2 - 2x \cdot c(x)}{x^4} = \frac{x(c'(x) \cdot x - 2c(x))}{x^4} =$

$$= \frac{c'(x)}{x^2} - \frac{2c(x)}{x^3}$$

$$y'_p + \frac{2}{x} \cdot y_p = x^3 \quad \bullet \text{ nonhomog. eq}$$

$$\frac{c'(x)}{x^2} - \frac{2c(x)}{x^3} + \frac{2c(x)}{x^3} = x^3 \Rightarrow c'(x) = x^5$$

$$\int c'(x) = \int x^5 dx = \frac{x^6}{6}$$

$$y_p = \frac{\cancel{c(x)} \cdot x^6}{6 \cdot x^2} = \frac{\cancel{c(x)} x^4}{6} = \frac{x^4}{6}$$

III $y = y_0 + y_p \Rightarrow y = x^{-2} \cdot c + \frac{x^4}{6}, \quad c \in \mathbb{R}$

c) $y' + 2x \cdot y = 2x \cdot e^{-x^2}$

$$\boxed{y = y_0 + y_p}$$

$$y' + 2x \cdot y = 0 \quad \text{homog eq}$$

$$y' = -2x \cdot y$$

$\underbrace{y}_{f(x)} \quad \underbrace{g(x)}$

$$y' = \frac{dy}{dx} = -2x \cdot y \Rightarrow \frac{dy}{y} = -2x dx \Rightarrow \int \frac{dy}{y} = \int -2x dx \Rightarrow \ln|y| = -2 \frac{x^2}{2} + C$$

$$\ln|y| = \ln e^{-x^2} \cdot c \Rightarrow y = e^{-x^2} \cdot c$$

$$y_p = e^{-x^2} \cdot c(x)$$

$$y'_p = -2x e^{-x^2} \cdot c(x) + c'(x) \cdot e^{-x^2}$$

$$y'_p + 2x \cdot y_p = 2x \cdot e^{-x^2} \Rightarrow -2x e^{-x^2} \cdot c(x) + c'(x) \cdot e^{-x^2} + 2x e^{-x^2} \cdot c(x) = 2x \cdot e^{-x^2}$$

$$c'(x) \cdot e^{-x^2} = 2x \cdot e^{-x^2} \Rightarrow c'(x) = 2x \Rightarrow c(x) = \int 2x dx = x^2$$

$$y_p = e^{-x^2} \cdot x^2$$

$$y = e^{-x^2} \cdot c + e^{-x^2} \cdot x^2, \quad c \in \mathbb{R}$$

$$f) y' + \frac{x}{1-x^2} \cdot y = x + \arcsin x$$

$$y' + \frac{x}{1-x^2} \cdot y = 0$$

$$y' = -\frac{x}{1-x^2} \cdot y$$

$$\frac{dy}{dx} = -\frac{x}{1-x^2} \cdot y \Rightarrow \frac{dy}{y} = dx \cdot \frac{-x}{1-x^2} \Rightarrow \int \frac{dy}{y} = \int dx \cdot \frac{-x}{1-x^2}$$

$$\ln|y| = \frac{1}{2} \ln|1-x^2| \cdot C$$

$$y = (1-x^2)^{\frac{1}{2}} \cdot C$$

$$y_0 = \sqrt{1-x^2} \cdot C$$

$$y_p = \sqrt{1-x^2} \cdot C(x)$$

$$y'_p = \frac{2x}{2\sqrt{1-x^2}} \cdot C(x) + C'(x) \cdot \sqrt{1-x^2}$$

$$\frac{x}{\sqrt{1-x^2}} C(x) + C'(x) \sqrt{1-x^2} + \frac{x}{1-x^2} \cdot \sqrt{1-x^2} \cdot C(x) = x + \arcsin x$$

$$C'(x) \sqrt{1-x^2} = x + \arcsin x$$

$$C'(x) = \frac{x + \arcsin x}{\sqrt{1-x^2}} \Rightarrow C(x) = \int \frac{x + \arcsin x}{\sqrt{1-x^2}} dx = \int \frac{x}{\sqrt{1-x^2}} dx + \int \arcsin x dx$$

$$= -\sqrt{1-x^2} + \arcsin x \cdot x + \sqrt{1-x^2}$$

Exercice 2

$$a) \begin{cases} xy' + y = e^x \\ y(a) = b, a, b \in \mathbb{R} \\ a \neq 0 \end{cases}$$

$$b) \begin{cases} y' + (2x^2 - 1)y = 2x^2 - 1 \\ y(1) = 1 - \frac{1}{e} \end{cases}$$

$$c) \begin{cases} y' - 2y = -x^2 \\ y(0) = \frac{1}{4} \end{cases}$$

$$\begin{cases} xy' + y = e^x \\ y(a) = b \end{cases} \quad \begin{matrix} a, b \in \mathbb{R} \\ a \neq 0 \end{matrix}$$

$$xy' + y = e^x$$

$$y' + \frac{y}{x} = \frac{e^x}{x}$$

$$y' + \frac{y}{x} = 0$$

$$y' = -\frac{y}{x}$$

$$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{-y} = \frac{dx}{x} \Rightarrow \ln|y| = \ln \frac{1}{x} \cdot c$$

$$y = \frac{c}{x}$$

$$y_0 = \frac{c}{x}$$

$$y_p = \frac{c(x)}{x}$$

$$y'_p = \frac{c'(x) \cdot x - c(x)}{x^2} = \frac{c'(x)}{x} - \frac{c(x)}{x^2}$$

$$\frac{c(x)}{x} + x \frac{c'(x)}{x} - \frac{c(x)}{x} = e^x \Rightarrow c'(x) = e^x$$

$$c(x) = e^x$$

$$y_p = \frac{e^x}{x}$$

$$y(x) = \frac{c}{x} + \frac{e^x}{x}, \quad c \in \mathbb{R}$$

$$y(a) = b \Rightarrow \frac{c + e^a}{a} = b \Rightarrow c + e^a = ab \Rightarrow c = ab - e^a$$

IVP SOL

$$y(x) = \frac{ab - e^a}{x} + \frac{e^x}{x}$$

$$b) \begin{cases} y' + (2x^2 - 1)y = 2x^2 - 1 \\ y(1) = 1 - \frac{1}{e} \end{cases}$$

$$\begin{cases} y' + (2x^2 - 1)(y - 1) = 0 \\ y(1) = 1 - \frac{1}{e} \end{cases}$$

$$\frac{dy}{dx} = -(2x^2 - 1)(y - 1) \Rightarrow \frac{dy}{y-1} = -\frac{dx}{2x^2 - 1}$$

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$$\int \frac{dx}{y-1} = -\int \frac{dx}{2x^2-1}$$

$$-\frac{1}{2x^2-1} = \frac{-1}{(\sqrt{2}x-1)(\sqrt{2}x+1)} = \frac{A}{(\sqrt{2}x-1)} + \frac{B}{(\sqrt{2}x+1)} = \frac{-1}{2(\sqrt{2}x-1)} + \frac{1}{2(\sqrt{2}x+1)}$$

$$-1 = A(\sqrt{2}x+1) + B(\sqrt{2}x-1)$$

$$-1 = \sqrt{2}x(A+B) + A - B \Rightarrow A+B=0 \quad A-B=-1 \Rightarrow A=-\frac{1}{2} \quad B=\frac{1}{2}$$

$$\frac{1}{\sqrt{2}}$$

g-

$$c) \begin{cases} y' - 2y = x^2 - x^2 \\ y(0) = \frac{1}{4} \end{cases}$$

$$y' - 2y = 0 \Rightarrow \frac{dy}{dx} = 2y \Rightarrow \frac{dy}{y} = 2dx \Rightarrow \ln|y| = \ln e^{2x} \cdot c \Rightarrow y = e^{2x} \cdot c$$

$$y_0 = e^{2x} \cdot c$$

$$y_p(x) = e^{2x} \cdot c(x)$$

$$y'_p(x) = 2e^{2x} \cdot c(x) + c'(x) \cdot e^{2x}$$

$$2e^{2x} \cdot c(x) + c'(x) \cdot e^{2x} - 2e^{2x} \cdot c(x) = -x^2$$

$$c'(x) \cdot e^{2x} = -x^2 \Rightarrow c'(x) = \frac{-x^2}{e^{2x}} \Rightarrow c(x) = \int \frac{-x^2}{e^{2x}} dx = \int x^2 \cdot e^{-2x} dx = \left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}\right)e^{-2x}$$

$$y_p(x) = e^{2x} \cdot e^{-2x} \left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}\right) = \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}$$

$$y = \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4} + e^{2x} \cdot c \quad c \in \mathbb{R}$$

$$y(0) = \frac{1}{4} \Rightarrow \frac{1}{4} = \frac{1}{4} + e^0 \cdot c \Rightarrow c = 0$$

$$y = \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}$$

? C=0?

THE BERNOULLI EQUATIONS

$$\text{General form: } y' + P(x) \cdot y = Q(x) \cdot y^\alpha \quad \alpha \neq 0 \text{ and } \alpha \neq 1$$

$$\text{subst: } z = y^{1-\alpha}$$

$$z(x) = y(x)^{1-\alpha}$$

$$\Downarrow$$
$$y = z^{\frac{1}{1-\alpha}} \Rightarrow y' = \frac{1}{1-\alpha} \cdot z^{\frac{1}{1-\alpha}-1} \cdot z'$$

$$y' = \frac{1}{1-\alpha} \cdot z^{\frac{\alpha}{1-\alpha}} \cdot z'$$

$$\Rightarrow \frac{1}{1-\alpha} \cdot z^{\frac{\alpha}{1-\alpha}} \cdot z' + P \cdot z^{\frac{1}{1-\alpha}} = Q \cdot z^{\frac{\alpha}{1-\alpha}} \quad | \cdot (1-\alpha) \cdot z^{-\frac{\alpha}{1-\alpha}} \Rightarrow$$

$$\Rightarrow z' + (1-\alpha)P \cdot z^{\frac{1}{1-\alpha}-\frac{\alpha}{1-\alpha}} = (1-\alpha)Q \Rightarrow$$

$$\Rightarrow \boxed{z' + (1-\alpha)P \cdot z = (1-\alpha)Q} \quad \text{nonhomogeneous linear equation}$$

$$\Rightarrow z = z_0 + z_p \Rightarrow y = (z_0 + z_p)^{\frac{1}{1-\alpha}}$$

Exercise 3:

$$a) x y' = 2x^2 \sqrt{y} + 4y$$

$$b) y' = y \cos x + y^2 \cos x$$

$$c) x y' + y + x^5 y^3 \cdot e^x = 0$$

$$d) y' + \frac{y}{x} = \frac{1}{x^2 y^2}$$

$$a) x y' = 2x^2 \sqrt{y} + 4y$$

$$y' - 2x \sqrt{y} - \frac{4y}{x} = 0$$

$$y' - \frac{4y}{x} = 2x y^{\frac{1}{2}}$$

$$\alpha = \frac{1}{2}$$

$$\text{subst: } z = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} \Rightarrow \boxed{y = z^2}$$

$$y(x) = z(x)^2$$

$$y'(x) = 2 \cdot z(x) \cdot z'(x)$$

$$2 \cdot z(x) \cdot z'(x) - \frac{4z^2(x)}{x} = 2x \cdot z(x) / 2z \Rightarrow z' - \frac{2}{x} z = x$$

linear

$$z' - \frac{2z}{x} = x$$

$$z' - \frac{2z}{x} = 0 \quad z' = \frac{2z}{x}$$

$$z' = \frac{dz}{dx} \Rightarrow \frac{dz}{dx} = \frac{2z}{x} \Rightarrow \frac{dz}{z} = dx \cdot \frac{2}{x} \Rightarrow \ln|z| = \ln|x|^2 + C$$

$$z = x^2 \cdot C$$

$$z_0(x) = C \cdot x^2, C \in \mathbb{R}$$

$$z_p(x) = x^2 \cdot c(x)$$

$$z'_p(x) = 2x \cdot c(x) + c'(x)x^2$$

$$2x \cdot c(x) + c'(x)x^2 - \frac{2}{x} x^2 c(x) = x$$

$$c'(x)x^2 = x \Rightarrow c'(x) = \frac{1}{x} \Rightarrow c(x) = \ln x$$

$$\Rightarrow z_p = x^2 \cdot \ln x$$

$$z = C \cdot x^2 + x^2 \cdot \ln x = x^2 (C + \ln x), C \in \mathbb{R}$$

$$y = z^2 \Rightarrow y(x) = x^4 (C + \ln x)^2, C \in \mathbb{R}$$

$$b) y' = y \cos x + y^2 \cos x \Rightarrow y' - y \cos x = y^2 \cos x$$

$$\alpha = 2$$

$$z = y^{1-2} \Rightarrow z = y^{-1} \Rightarrow y = \frac{1}{z}$$

$$y' = -\frac{1}{z^2} \cdot z'$$

$$-\frac{1}{z^2} \cdot z' - \frac{1}{z} \cos x = \cos x \cdot \frac{1}{z^2} \quad / \cdot z^2$$

$$z' + z \cos x = -\cos x$$

$$z' + z \cos x = 0$$

$$z' = -z \cos x$$

$$\frac{dz}{dx} = -z \cos x \Rightarrow \frac{dz}{z} = dx \cdot \cos x \Rightarrow \ln z = -\sin x + C$$

$$z = e^{-\sin x} \cdot C$$

$$z_0 = e^{-\sin x} \cdot C(x)$$

$$z_p = e^{-\sin x} \cdot c(x)$$

$$z'_p = -\cos x \cdot e^{-\sin x} \cdot c(x) + c'(x) e^{-\sin x}$$

$$-\cos x \cdot e^{-\sin x} \cdot c(x) + c'(x) e^{-\sin x} + e^{-\sin x} \cdot \cos x = -\cos x$$

$$c'(x) e^{-\sin x} = -\cos x \Rightarrow c'(x) = \frac{-\cos x}{e^{-\sin x}} = -\cos x \cdot e^{\sin x}$$

$$C(x) = \int -\cos x \cdot e^{\sin x} dx = -e^{\sin x}$$

$$z_p = -e^{nimx} \cdot e^{-nimx} = -1$$

$$z = z_0 + z_p = e^{-nimx} \cdot C - 1, C \in \mathbb{R}$$

$$z = y^{-1} \Rightarrow y = \frac{1}{C \cdot e^{-nimx} - 1}, C \in \mathbb{R}$$

$$c) xy' + y + x^5 y^3 \cdot e^x = 0$$

$$y' + xy + x^4 y^3 \cdot e^x = 0$$

$$y' + xy = -x^4 y^3 \cdot e^x$$

$$\alpha = 3$$

$$z = y^{1-3} = y^{-2} \Rightarrow y = z^{-\frac{1}{2}} = \frac{1}{\sqrt{z}}$$

$$y' = \frac{1}{\sqrt{z}} \cdot \left(-\frac{1}{2} z^{-\frac{3}{2}} \cdot z' \right) = -\frac{1}{2} \cdot z^{-\frac{3}{2}} \cdot z'$$

$$-\frac{1}{2} \cdot z^{-\frac{3}{2}} \cdot z' + z^{-\frac{1}{2}} = x^4 \cdot \frac{1}{z} \cdot e^x \quad / \cdot z^{\frac{3}{2}}$$

$$-\frac{1}{2} \cdot z' + z^{-\frac{1}{2}} x = x^4 \cdot e^x$$

$$z' - 2z^{-\frac{1}{2}} x = -2x^4 \cdot e^x$$

$$z' - 2z^{-\frac{1}{2}} x = 0$$

$$z' = 2z^{-\frac{1}{2}} x$$

$$\frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$