

SOLVABLE DIFFERENTIAL EQUATIONS IN IMPLICIT FORM

$$F(x, y, y') = 0$$

1) THE CLAIREAUT DIFF. EQ

the general form: $y = x \cdot y' + \Psi(y')$, $\Psi \in C^1$

- we derivate the eq. with respect to x

$$y' = y' + x \cdot y'' + \Psi'(y') \cdot y''$$

$$y''(x + \Psi'(y')) = 0$$

- we derivate by $\boxed{p = y'}$

$$p' \cdot (x + \Psi'(p)) = 0 \rightarrow p' = 0 \Rightarrow \boxed{p = c}, c \in \mathbb{R}$$

$$\rightarrow x + \Psi'(p) = 0 \Rightarrow$$

$$\Rightarrow y(x) = x \cdot c + \Psi(c), c \in \mathbb{R} \quad \text{the general solution}$$

$$x + \Psi(p) = 0 \Rightarrow \begin{cases} x = -\Psi'(p) \\ y = x \cdot p + \Psi(p) \end{cases}$$

$$y' = p$$

$$\begin{matrix} \uparrow \\ -\Psi'(p) \end{matrix}$$

the singular solution in parametric form

Exercise 1:

a) $y = x y' - (y')^2$

b) $y = x y' + \sqrt{1 + y'^2}$

c) $y = x y' + \frac{1}{5} (y')^5$

$$a) \underline{y = xy' - \underbrace{(y')^2}_{\psi(y')}} \quad \psi(y') = -(y')^2$$

$$\psi(y') = -(y')^2$$

- we derivate the eq. with respect to x.

$$\Rightarrow y' = y' + x \cdot y'' - 2y' \cdot y'' \Rightarrow$$

$$\Rightarrow y''(x - 2y') = 0$$

$$\underline{\text{denote } y' = p \Rightarrow p'(x - 2p) = 0 \Rightarrow}$$

$$p' = 0 \Rightarrow p = c, c \in \mathbb{R} \Rightarrow$$

$$y(x) = x \cdot c - c^2, c \in \mathbb{R}$$

the general solution

$$x - 2p = 0 \Rightarrow \begin{cases} x = 2p \\ y = xp - p^2 \end{cases}$$

$$\begin{cases} x = 2p \\ y = 2p^2 - p^2 \end{cases}$$

$$\begin{cases} x = 2p \\ y = p^2 \end{cases} \text{ the singular sol. in parametric form}$$

$$x = 2p \Rightarrow p = \frac{x}{2} \Rightarrow y(x) = \left(\frac{x}{2}\right)^2$$

$$y(x) = \frac{x^2}{4}$$

the singular solution in explicit form

$$b) \underline{y = xy' + \sqrt{1+y'^2}} \quad \psi(y') = \sqrt{1+y'^2}$$

$$\psi(y') = \sqrt{1+y'^2}$$

- we derivate

$$y' = y' + xy'' + \frac{1}{2\sqrt{1+y'^2}} \cdot 2y' \cdot y''$$

$$0 = xy'' + \frac{y' \cdot y''}{\sqrt{1+y'^2}} \Rightarrow y'' \left(x + \frac{y'}{\sqrt{1+y'^2}} \right) = 0$$

$$\text{- denote } y' = p \Rightarrow p' \left(x + \frac{p}{\sqrt{1+p^2}} \right) = 0$$

$$\bullet p' = 0 \Rightarrow p = c \in \mathbb{R} \Rightarrow y' = c \xrightarrow{y = xy' + \sqrt{1+y'^2}}$$

$$y = xc + \sqrt{1+c^2}, c \in \mathbb{R}$$

general solution

$$\bullet x + \frac{p}{\sqrt{1+p^2}} = 0 \Rightarrow x = -\frac{p}{\sqrt{1+p^2}}$$

$$x = -\frac{p}{\sqrt{1+p^2}}$$

$$y = xp + \sqrt{1+p^2}$$

$$\Rightarrow y = \frac{-p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2} \Rightarrow y = \frac{-p^2 + 1 + p^2}{\sqrt{1+p^2}} = \frac{1}{\sqrt{1+p^2}}$$

$$\begin{cases} x = \frac{-p}{\sqrt{1+p^2}} \\ y = \frac{1}{\sqrt{1+p^2}} \end{cases}$$

the singular solution in parametric form

show that $x^2 + y^2 = \left(\frac{-p}{\sqrt{1+p^2}}\right)^2 + \left(\frac{1}{\sqrt{1+p^2}}\right)^2 = \frac{p^2+1}{1+p^2} = 1$

$x^2 + y^2 = 1$ singular solution in implicit form

$y = \pm \sqrt{1-x^2}$ singular solution in explicit form

? c) $y = xy' + \frac{1}{5} y'^5$

$\psi(y') = \frac{1}{5} y'^5$

derivate

$$y' = y' + xy'' + \frac{5}{5} y'^4 \cdot y'' \Rightarrow 0 = xy'' + y'^4 \cdot y''$$

$$0 = y''(x + y'^4)$$

denote $y' = p \Rightarrow p'(x + p^4) = 0$

$p' = 0 \Rightarrow p = c, c \in \mathbb{R} \rightarrow y' = c \xrightarrow{y = xy' + \frac{1}{5} y'^5} y = xc + \frac{1}{5} c^5, c \in \mathbb{R}$ general solution

$x + p^4 = 0 \Rightarrow x = -p^4$

$$\begin{cases} y = -p^4 \cdot p + \frac{1}{5} p^5 \\ x = -p^4 \end{cases}$$

$$\begin{cases} y = -\frac{4}{5} p^5 \\ x = -p^4 \end{cases}$$

singular sol. in parametric form

$y = -\frac{4}{5} p^5$

$x = -p^4 \Rightarrow$

2) THE LAGRANGE DIFFERENTIAL EQUATIONS

the general form: $y = x \varphi(y') + \psi(y')$ $\varphi, \psi \in C^1$

- We derivate the equation w.r.t. x

- we denote $y' = p \Rightarrow$

$$\Rightarrow \begin{cases} x(p) = \dots \\ y(p) = x(p) \cdot \varphi(p) + \psi(p) \end{cases}$$

the general solution in parametric form

Exercise 2

a) $y = x(1+y') + (y')^2$

b) $y = \frac{3}{2}x y' + e^{y'}$

c) $y = x(y')^2 - \frac{1}{y'}$

a) $y = x(1+y') + (y')^2$

$$\varphi(y') = 1+y' \quad \psi(y') = (y')^2$$

derivate the eq. with respect to x

$$y' = (1+y') + x y'' + 2y' \cdot y''$$

$$0 = 1 + y''(x + 2y')$$

$$\text{denote } y' = p \Rightarrow 1 + p'(x + 2p) = 0 \Rightarrow p'(x + 2p) = -1$$

$$p = p(x) \quad p' = p'(x) = \frac{dp}{dx} \Rightarrow \frac{dp}{dx}(x + 2p) = -1 \quad / \cdot \frac{dx}{dp}$$

$$x + 2p = - \underbrace{\frac{dx}{dp}}_{x'(p)} \Rightarrow \text{the unknown function } x = x(p)$$

$$x + 2p = x'$$

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$$x + 2p = x'$$

$$x(p) + 2p = -x'(p) \Rightarrow x'(p) + x p = -2p \quad \text{nonhom. first order linear diff. eq.}$$

$$x' + x = -2p$$

$$x' + x = 0 \Rightarrow x' = -x \Rightarrow \frac{dx}{dp} = -x \Rightarrow \frac{dx}{-x} = dp \Rightarrow \int \frac{dx}{-x} = \int dp \Rightarrow \ln x = -p + \ln c$$

$$\ln x = \ln c e^{-p}$$

$$x_0(p) = c \cdot e^{-p}, c \in \mathbb{R}$$

$$x_p = c(p) \cdot e^{-p}$$

$$x'_p = c'(p) \cdot e^{-p} - c(p) \cdot e^{-p}$$

$$x'_p + x_p = -2p \Rightarrow c(p) \cdot e^{-p} + c'(p) \cdot e^{-p} - c(p) \cdot e^{-p} = -2p \Rightarrow$$

$$\Rightarrow c'(p) \cdot e^{-p} = -2p \Rightarrow c'(p) = -2p \cdot e^p \Rightarrow c(p) = -2 \int p \cdot e^p dp$$

$$\int p \cdot e^p dp = p \cdot e^p - \int e^p dp = p \cdot e^p - e^p$$

$$f(x) = p \quad f'(x) = 1$$

$$g(x) = e^p \quad g'(x) = e^p$$

$$c(p) = -2(p \cdot e^p - e^p) = -2e^p(p - 1)$$

$$x_p = 2e^p(p - 1) \cdot e^{-p} = -2p + 2$$

$$x = x_p + x_0 = -2p + 2 + c \cdot e^{-p}, c \in \mathbb{R}$$

$$y' = p \frac{x(1+y') + (y')^2}{(1+y')^2} \Rightarrow y = x(1+p) + p^2$$

$$\begin{cases} x(p) = x(1+y') + (y')^2 \\ y(p) = (c \cdot e^{-p} - 2p + 2)(1+p) + p^2, c \in \mathbb{R} \end{cases}$$

the general solution in parametric form

Remark: if there exist $\alpha \in \mathbb{R}$ pt. $\varphi(\alpha) = \alpha$ then $y(x) = \alpha \cdot x + \varphi(\alpha)$ is a singular solution of the Lagrange equation.

$$\varphi(\alpha) = 1 + \alpha$$

$$\varphi(\alpha) = \alpha$$

$$1 + \alpha = \alpha$$

$$1 = 0 \quad \text{F}$$

$\varphi(\alpha) = \alpha$ has no solution in $\mathbb{R} \Rightarrow$ the diff. eq. has no singular solution

$$b) y = \frac{3}{2}xy + e^{y'}$$

$$f(y') = \frac{3}{2}y' \quad \psi(y') = e^{y'}$$

$$f(\alpha) = \alpha \Rightarrow \alpha = \frac{3}{2}\alpha \Rightarrow \alpha - \frac{3}{2}\alpha = 0 \Rightarrow +\frac{1}{2}\alpha = 0 \Rightarrow$$

$$\Rightarrow y(x) = \frac{3}{2}x \cdot 0 + e^0 = 1 \text{ singular sol. of the eq.}$$

1. derivate w.r.t. x

$$y' = \frac{3}{2}y' + \frac{3}{2}xy'' + y'' \cdot e^{y'}$$

$$0 = \frac{1}{2}y' + y''(\frac{3}{2}x + e^{y'})$$

2. denote $y' = p \Rightarrow 0 = \frac{1}{2}p + p''(\frac{3}{2}x + e^p)$

$$p' = \frac{dp}{dx} \Rightarrow \frac{dx}{dp} = x'(p)$$

$$0 = \frac{1}{2}p + p'(\frac{3}{2}x + e^p) \Rightarrow -\frac{1}{2}p = \frac{dp}{dx}(\frac{3}{2}x + e^p) \quad / \cdot \frac{dx}{dp}$$

$$-\frac{1}{2}p \cdot x'(p) = \frac{3}{2}x + e^p$$

$$-e^p = \frac{3}{2}x + \frac{1}{2}p x'(p) \quad \text{nonhomogeneous linear diff. eq.}$$

$$x'(p) + \frac{3x}{p} = \frac{-2e^p}{p}$$

$$x'(p) + \frac{3x}{p} = 0 \Rightarrow x'(p) = -\frac{3x}{p} \Rightarrow \frac{dx}{dp} = -\frac{3x}{p} \Rightarrow \frac{dx}{x} = -\frac{3}{p} \frac{dp}{p} \Rightarrow$$

$$\Rightarrow \ln x = -3 \ln p + \ln c \Rightarrow \ln x = \ln p^{-3} \cdot c \Rightarrow x = p^{-3} \cdot c, c \in \mathbb{R}$$

$$x_0 = p^{-3} c$$

$$\begin{cases} xp = p^{-3} \cdot c(p) \\ x' = -3 \cdot p^{-4} \cdot c(p) + c'(p) \cdot p^{-3} \end{cases}$$

$$\Rightarrow -3 \cdot p^{-4} \cdot c(p) + c'(p) \cdot p^{-3} + \frac{3p^{-3} \cdot c(p)}{p} = \frac{-2e^p}{p} \Rightarrow$$

$$\Rightarrow c'(p) \cdot p^{-3} = \frac{-2e^p}{p} \Rightarrow c'(p) = -2e^p \cdot p^2 \Rightarrow c(p) = \int -2e^p \cdot p^2 dp =$$

$$= -2p^2 \cdot e^p - \int -4p \cdot e^p dp = -2p^2 \cdot e^p + 4(p e^p - e^p) = e^p(-2p^2 + 4p - 4)$$

$$f(x) = p^2$$

$$f'(x) = 2p$$

$$g(x) = e^p$$

$$g(x) = e^p$$

$$f_2(x) = p$$

$$f_2'(x) = 1$$

$$g_1(x) = e^p$$

$$g(x) = e^p$$

$$x_p = p^{-3} \cdot e^p(-2p^2 + 4p - 4) = e^p(-\frac{2}{p} + \frac{4}{p^2} - \frac{4}{p^3})$$

$$x = x_p + x_0 = p^{-3} \cdot c + e^p(-\frac{2}{p} + \frac{4}{p^2} + \frac{1}{p^3}), c \in \mathbb{R}$$

$$y = \frac{3}{2} x y' + e^{y'} \Rightarrow y(p) = \frac{3}{2} (p^{-3} \cdot C + e^p (-\frac{2}{p} + \frac{4}{p^2} + \frac{4}{p^3})) \cdot p + e^p$$

$\uparrow$ $\uparrow$
 $x(p)$ p

general solution in parametric form

$$c) y = x(y')^2 - \frac{1}{y'}$$

$$\varphi(y') = (y')^2 \quad \psi(y') = -\frac{1}{y'}$$

$$\varphi(\alpha) = \alpha \Rightarrow \alpha^2 = \alpha \Rightarrow \alpha(\alpha-1) = 0 \Rightarrow \alpha = 0 \vee \alpha = 1$$

singular sol. of the equation

1. derivate w.r.t. x

$$y' = 1(y')^2 + 2y' \cdot y''x + \frac{1}{(y')^2} \cdot y'' \quad \boxed{y = x-1}$$

$$1 = y' + 2 \cdot y''x + \frac{y''}{(y')^3}$$

$$1 = y' + y''(2x + \frac{1}{(y')^3})$$

$$1 - y' = y''(2x + \frac{1}{(y')^3})$$

$$p = p(x)$$

$$y = y(x)$$

$$\text{Denote } p = y' \Rightarrow 1 - p = p'(2x + \frac{1}{p^3})$$

$$p' = \frac{dp}{dx} \Rightarrow 1 - p = \frac{dp}{dx} (2x + \frac{1}{p^3}) \quad \bigg/ \frac{dx}{dp}$$

$$\frac{dx}{dp} (1-p) = 2x + \frac{1}{p^3}$$

$$x'(1-p) = 2x + \frac{1}{p^3}$$

$$x' - \frac{2}{1-p} x = \frac{1}{p^3(1-p)}$$

linear differential equation

$$x' - \frac{2}{1-p} x = 0$$

$$x' = \frac{2}{1-p} x \Rightarrow \frac{dx}{dp} = \frac{2}{1-p} x \Rightarrow \frac{dx}{x} = \frac{2 dp}{1-p} \Rightarrow \ln x = -2 \ln(1-p) + \ln C \Rightarrow$$

$$\Rightarrow \ln x = \ln(1-p)^{-2} \cdot C \Rightarrow x = \frac{1}{(1-p)^2} \cdot C$$

$$x_0 = \frac{1}{(1-p)^2} \cdot C$$

$$x_p = \frac{1}{(1-p)^2} \cdot C(p)$$

$$x'p = \frac{C'(p) \cdot (1-p)^2 + 2(1-p) \cdot C(p)}{(1-p)^4} = \frac{(1-p)[C'(p)(1-p) + 2C(p)]}{(1-p)^4} = \frac{C'(p)(1-p) + 2C(p)}{(1-p)^3}$$

$$\frac{C'(p)}{(1-p)^3} + \frac{2C(p)}{(1-p)^3} - \frac{2C(p)}{(1-p)^3} = \frac{1}{p^3(1-p)} \Rightarrow \frac{C'(p)}{(1-p)^2} = \frac{1}{p^3(1-p)} \Rightarrow C'(p) = \frac{1-p}{p^3} = \frac{1}{p^3} - \frac{1}{p^2}$$

$$C(p) = \int \frac{1}{p^3} dp - \int \frac{1}{p^2} dp = \frac{-1}{2p^2} + \frac{1}{p}$$

$$X_p = \frac{1}{(1-p)^2} \cdot \left(-\frac{1}{2p^2} + \frac{1}{p} \right)$$

$$X = X_p + X_0 = \frac{1}{(1-p)^2} \left(\frac{1}{p} - \frac{1}{2p^2} \right) + \frac{c}{(1-p)^2} = \frac{\left(\frac{1}{p} - \frac{1}{2p^2} + c \right)}{(1-p)^2}, c \in \mathbb{R}$$

$$y = X(y')^2 - \frac{1}{y'} \Rightarrow y = Xp^2 - \frac{1}{p} \Rightarrow y(p) = \frac{\frac{1}{p} - \frac{1}{2p^2} + c}{(1-p)^2} \cdot p^2 - \frac{1}{p} \Rightarrow$$

$$\Rightarrow y(p) = \frac{p - \frac{1}{2} + p^2 c}{(1-p)^2} - \frac{1}{p} = \frac{2p - 1 + 2p^2 c}{2(1-p)^2}, c \in \mathbb{R}$$

general solution in parametric form

3. THE EXACT DIFFERENTIAL EQUATION

the general form: $g(x, y) + h(x, y) \cdot y' = 0$

$$y' = \frac{dy}{dx}$$

$$g(x, y) + h(x, y) \cdot y' = 0 \quad /$$

$$g(x, y) + h(x, y) \cdot \frac{dy}{dx} = 0 \quad / \cdot dx$$

$$g(x, y) dx + h(x, y) \cdot dy = 0$$

$$u = u(x, y)$$

$$du = \frac{\partial u}{\partial x}(x, y) \cdot dx + \frac{\partial u}{\partial y}(x, y) \cdot dy$$

Exact diff eq. if $\exists u(x, y)$, $u \in C^1$, s.t.

$$g(x, y) = \frac{\partial u}{\partial x}(x, y)$$

$$h(x, y) = \frac{\partial u}{\partial y}(x, y)$$

The condition that a diff. eq. is an exact differential equation

$$\text{iff: } \frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}$$

+EXERCITII : incomplet