

SEMINAR 4 ECUATII

SOLVABLE HIGHER ORDER DIFFERENTIAL EQUATIONS

$$y^{(m)} = f(x, y, y', \dots, y^{(m-1)}), \quad m \geq 2$$

I Diff. eq. of the form: $y^{(m)} = f(x)$

f is continuous function

$$y^{(m)} = f(x) \Rightarrow (y^{(m-1)})' = f(x)$$

$$\Rightarrow y^{(m-1)}(x) = \int f(x) dx + c_1, \quad c_1 \in \mathbb{R}$$

$$y^{(m-1)}(x) = F(x) + c_1$$

$$(y^{(m-2)}(x))' = F(x) + c_1$$

$$y^{(m-2)}(x) = \int (F(x) dx + c_1) dx + c_2$$

$$= \int F(x) dx + x c_1 + c_2$$

$$= F_2(x) dx + x c_1 + c_2$$

...

Ex1

a) $y''' = x + \cos x + \sin x$

b) $y''' = \ln x$

c) $y'' = x \cdot e^x$

d) $y'' = 1 + \tan^2 x$

$$a) y''' = x + \cos x + \sin x$$

$$(y'')' = x + \cos x + \sin x$$

$$y'' = \int x + \cos x + \sin x \, dx + c_1 = \frac{x^2}{2} - \cos x + \sin x + c_1$$

$$(y')' = \frac{x^2}{2} - \cos x + \sin x + c_1$$

$$y' = \int \frac{x^2}{2} - \cos x + \sin x \, dx + c_1 + c_2 = \frac{x^3}{6} - \sin x - \cos x + x c_1 + c_2$$

$$y(x) = \int \left(\frac{x^3}{6} - \sin x - \cos x + x c_1 + c_2 \right) dx + c_3 = \frac{x^4}{24} + \cos x - \sin x + \frac{x^2}{2} c_1 + x c_2 + c_3,$$

$$c_1, c_2, c_3 \in \mathbb{R}$$

$$b) y''' = \ln x$$

$$\begin{aligned} (y'')' = \ln x &\Rightarrow y'' = \int \ln x \, dx + c_1 = \int (x)' \cdot \frac{1}{x} dx + c_1 - \\ &= x \ln x - \int x \cdot \frac{1}{x} dx + c_1 = \\ &= x \ln x - x + c_1 \end{aligned}$$

$$(y')' = x \ln x - x + c_1$$

$$y' = \int (x \ln x - x + c_1) dx + c_2 = \int \frac{x^2}{2} \ln x \, dx - \int x \, dx + \int c_1 \, dx + c_2 =$$

$$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx - \frac{x^2}{2} + x c_1 + c_2 =$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} - \frac{x^2}{2} + x c_1 + c_2 =$$

$$= \frac{x^2}{2} \ln x - \frac{3}{4} x^2 + x c_1 + c_2$$

$$y(x) = \int \left(\frac{x^2}{2} \ln x - \frac{3}{4} x^2 + x c_1 + c_2 \right) dx + c_3 = \int \frac{x^2}{2} \ln x \, dx - \frac{3}{4} \int x^2 \, dx + \int x c_1 \, dx + \int c_2 \, dx + c_3 =$$

$$= \frac{x^3}{6} \ln x - \int \frac{x^2}{6} \cdot \frac{1}{x} dx - \frac{3}{4} \frac{x^3}{3} + \frac{x^2}{2} c_1 + x c_2 + c_3 =$$

$$= \frac{x^3}{6} \ln x - \frac{x^3}{18} - \frac{x^3}{4} + \frac{x^2}{2} c_1 + x c_2 + c_3 =$$

$$= \frac{x^3}{6} \ln x - \frac{11x^3}{36} + \frac{x^2}{2} c_1 + x c_2 + c_3, \quad c_1, c_2, c_3 \in \mathbb{R}$$

$$c) y'' = x \cdot e^x$$

$$(y')' = x \cdot e^x \Rightarrow y' = \int x \cdot e^x dx + c_1 = e^x x - e^x + c_1$$

$$y(x) = \int e^x \cdot x - e^x + c_1 dx + c_2 = e^x x - e^x - e^x + x c_1 + c_2 = e^x x - 2e^x + x c_1 + c_2, \quad c_1, c_2 \in \mathbb{R}$$

$$d) y'' = 1 + \tan^2 x$$

$$(y')' = 1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Rightarrow y' = \int \frac{1}{\cos^2 x} dx + c_1 = \tan x + c_1$$

$$y(x) = \int \tan x + c_1 dx + c_2 = -\ln |\cos x| + c_1 x + c_2$$

II. Differential equations of the form $F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(m)}) = 0, k \geq 2$

subst $y^{(k)} = z \Rightarrow F(x, z, z', \dots, z^{(m-k)}) = 0$ differential equation of order $m-k$

if the $m-k$ order diff. eq. is SOLVABLE and the solution is obtained in explicit form $z(x) = \varphi(x, c_1, \dots, c_{m-k}) \Rightarrow y^{(k)} = z \Rightarrow y^{(k)}(x) = \varphi(x, c_1, \dots, c_{m-k})$ k order diff. eq. of type 1

$$y(x) = \psi(x, c_1, \dots, c_m)$$

Ex2

$$a) y^{(4)} + y''' = 0$$

$$b) xy''' - 3y'' - 4x^2 = 0$$

$$c) y'(1+y') = a \cdot y'', a \in \mathbb{R}$$

$$d) xy'' + y' + x = 0$$

$$a) y^{(4)} + y''' = 0$$

$$\boxed{z = y'''} \Rightarrow \underline{z' + z = 0} \quad \text{first order linear homogeneous diff eq.}$$

$$z' = -z \Rightarrow \frac{dz}{dx} = -z \Rightarrow \frac{dz}{z} = -x dx \Rightarrow \ln z = \ln e^{-x} \cdot c, c \in \mathbb{R}$$

$$z = e^{-x} \cdot c$$

$$y''' = z \Rightarrow y''' = c_1 e^{-x} \Rightarrow (y'')' = c_1 e^{-x} \Rightarrow$$

$$y'' = \int c_1 e^{-x} dx + c_2 = -c_1 e^{-x} + c_2$$

$$(y')' = -c_1 e^{-x} + c_2 \Rightarrow$$

$$y' = \int -c_1 e^{-x} + c_2 dx + c_3 = c_1 e^{-x} + c_2 x + c_3$$

$$y(x) = \int c_1 e^{-x} + c_2 x + c_3 dx + c_4 = -c_1 e^{-x} + c_2 \frac{x^2}{2} + c_3 x + c_4,$$

$$c_1, c_2, c_3, c_4 \in \mathbb{R}$$

$$b) xy''' - 3y'' - 4x^2 = 0$$

$$\text{subst } z = y'' \Rightarrow xz' - 3z = 4x^2 \quad \text{first order linear nonhomogeneous equation}$$

$$\boxed{z' - \frac{3}{x}z = 4x}$$

$$z' - \frac{3}{x}z = 4x$$

$$z' - \frac{3}{x}z = 0 \Rightarrow z' = \frac{3}{x}z \Rightarrow \frac{dz}{dx} = \frac{3z}{x} \Rightarrow \frac{dz}{z} = 3 \frac{dx}{x} \Rightarrow \ln z = \ln x^3 \cdot c \Rightarrow$$

$$\Rightarrow z_0 = x^3 \cdot c$$

$$z_p = x^3 \cdot c(x)$$

$$z'_p = 3x^2 \cdot c(x) + c'(x) x^3$$

$$3x^2 \cdot c(x) + c'(x) x^3 - 3x^2 \cdot c(x) = 4x$$

$$c'(x) x^3 = 4x \Rightarrow c'(x) = \frac{4}{x^2} \Rightarrow c(x) = \int \frac{4}{x^2} dx = -\frac{4}{x}$$

$$\begin{cases} z_p = x^3 \cdot c(x) \\ c(x) = -\frac{4}{x} \end{cases} \Rightarrow z_p = x^3 \cdot \frac{(-4)}{x} = -4x^2$$

$$z = z_p + z_0 = -4x^2 + x^3 \cdot c, c \in \mathbb{R}$$

$$z = y'' \Rightarrow y'' = -4x^2 + x^3 \cdot c$$

$$(y')' = -4x^2 + x^3 c \Rightarrow y' = \int -4x^2 + x^3 c dx + c_1 = -\frac{4x^3}{3} + \frac{x^4}{4} c + c_1 \quad c, c_1, c_2 \in \mathbb{R}$$

$$y' = -\frac{4x^3}{3} + \frac{x^4}{4} c + c_1 \Rightarrow y = \int -\frac{4x^3}{3} + \frac{x^4}{4} c + c_1 dx + c_2 = -\frac{4x^4}{12} + \frac{x^5}{20} c + x c_1 + c_2$$

$$c) y'' - \frac{(y')^2}{y} - y y' = 0$$

$$y' = p \Rightarrow y'' = p' \cdot y' = p' \cdot p$$

$$p' \cdot p - \frac{p^2}{y} - y p = 0$$

$$p' - \frac{p}{y} - y = 0$$

$$\boxed{p' - \frac{p}{y} = y} \quad \text{non homogeneous linear equation}$$

$$p' - \frac{p}{y} = 0 \Rightarrow p' = \frac{p}{y} = \dots = p_0 = c \cdot y$$

$$p_0 = c(y) \cdot y$$

$$p'_p = c'(y) \cdot y + c(y)$$

$$c'(y) \cdot y + c(y) - c(y) = y$$

$$c'(y) = 1 \Rightarrow c(y) = y$$

$$p_0 = y^2$$

$$p = p_0 + p_p = c \cdot y + y^2$$

$$y' = c \cdot y + y^2$$

separable

$$c \cdot y + y^2 = 0 \Rightarrow y(c + y) = 0 \Rightarrow y = 0 \text{ or } y = -c \quad y(x) = 0 \text{ singular sol}$$

$$\frac{dy}{dx} = c \cdot y + y^2 \Rightarrow \frac{dy}{c \cdot y + y^2} = dx \Rightarrow \int \frac{dy}{c \cdot y + y^2} = \int dx \Rightarrow$$

$$\begin{aligned} \int \frac{dy}{c \cdot y + y^2} &= \int \frac{dy}{y(c + y)} = \int \frac{1}{y \cdot c} dy + \int \frac{-1}{c(c + y)} dy = \frac{1}{c} \ln y - \frac{1}{c} \ln(c + y) = \\ &= \frac{1}{c} (\ln y - \ln(c + y)) = \\ &= \frac{1}{c} \left(\ln \frac{y}{c + y} \right) \end{aligned}$$

$$\frac{1}{c} \ln \frac{y}{c + y} = x + C_1 \Rightarrow \ln \frac{y}{c + y} = (x + C_1) c \Rightarrow \frac{y}{c + y} = e^{(x + C_1) c} \Rightarrow$$

$$\Rightarrow \frac{y + c}{c + y} - \frac{c}{c + y} = e^{(x + C_1) c} \Rightarrow$$

$$\frac{-c}{c + y} = e^{(x + C_1) \cdot c} - 1 \quad \text{/} \quad -\frac{1}{c}$$

$$\frac{1}{c + y} = -\frac{1}{c} \cdot (e^{(x + C_1) \cdot c} - 1)$$

$$c + y = \frac{-c}{e^{(x + C_1) \cdot c} - 1} \quad \text{/} \quad -c$$

$$y = \frac{-c}{e^{(x + C_1) \cdot c} - 1} - c \quad \text{general solution}$$