SEMINAR 5
ECUAȚII

PROBLEMS ATTACHED TO DIFFERENTIAL EQUATIONS

INITIAL VALUE PROBLEM (CAUCHY PROBLEM) IVP

In general IVP for an n-order diff. eq.:

$$\begin{cases} y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

- find general solution of the diff eq.
- using the initial conditions we find the values of the integration constants
- replace the constants in gen. sol. expression $\Rightarrow$ i.v.p. sol

1. Find the following IVP solutions:

a) $\begin{cases} (1+e^x) y \cdot y' - e^x = 0 \\ y(0) = 1 \end{cases}$

b) $\begin{cases} y' \cdot \sin x - y \ln y = 0 \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$

c) $\begin{cases} x^2 y' = y^2 - xy \\ y(-1) = -1 \end{cases}$

d) $\begin{cases} xy' + y = e^x \\ y(a) = b \end{cases}$ linear eq. $a, b \in \mathbb{R}, a \neq 0$

e) $\begin{cases} y' + \frac{y}{x} = \frac{1}{x^2 y^2} \\ y(1) = 0 \end{cases}$ Bernoulli eq

a) $\begin{cases} (1+e^x) y \cdot y' - e^x = 0 \\ y(0) = 1 \end{cases} \Rightarrow (1+e^x) y \cdot y' = e^x \Rightarrow y' = \frac{e^x}{(1+e^x) \cdot y} \Rightarrow$ separable diff eq.

$$y' = \underbrace{\frac{e^x}{1+e^x}}_{f(x)} \cdot \underbrace{\frac{1}{y}}_{g(y)}$$

$g(y) = 0 \Rightarrow \frac{1}{y} = 0 \Rightarrow$ no sol

$$\frac{dy}{dx} = \frac{e^x}{1+e^x} \cdot \frac{1}{y} \Rightarrow y\,dy = \frac{e^x}{1+e^x}\,dx \Rightarrow 2\int y\,dy = \int \frac{2e^x}{e^x+1}\,dx \Rightarrow$$

$$\Rightarrow y^2 = 2\ln(e^x+1) + C$$

$$y^2 = 2\ln(e^x+1) + C$$

$$\boxed{y = \pm\sqrt{2\ln(e^x+1)+C}}, \quad C \in \mathbb{R}$$

general solution

$$y(0) = 1 \Rightarrow y(x) = \pm\sqrt{2\ln(e^x+1)+C}$$

$$1 = \pm\sqrt{2\ln(1+1)+C}$$

$$1 = \pm\sqrt{2\ln 2 + C}$$

$$1 = 2\ln 2 + C \Rightarrow C = 1 - 2\ln 2$$

$$\Rightarrow \text{IVP sol} \quad y(x) = \sqrt{2\ln(e^x+1)+1-2\ln 2}$$

b) $$\begin{cases} y' \cdot \sin x - y\ln y = 0 \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$$

$$y' \cdot \sin x = y\ln y \Rightarrow y' = \frac{y\ln y}{\sin x} = \underbrace{y\ln y}_{g(y)} \cdot \underbrace{\frac{1}{\sin x}}_{f(x)}$$

separable diff. eq.

$g(y) = 0 \Rightarrow y\ln y = 0 \nearrow y = 0$ not convenient

$\searrow \ln y = 0 \Rightarrow y = 1$

$\boxed{y(x) = 1}$ singular solution

$$\frac{dy}{dx} = y\ln y \cdot \frac{1}{\sin x} \Rightarrow \frac{dy}{y\ln y} = \frac{dx}{\sin x} \Rightarrow \int \frac{dy}{y\ln y} = \int \frac{dx}{\sin x} \Rightarrow$$

$$\int \frac{dy}{y\ln y} = \int \frac{1}{y} \cdot \frac{1}{\ln y}\,dy = \int \frac{1}{t}\,dt = \ln t = \ln(\ln y)$$

N: $\ln y = t$

D: $\frac{1}{y}dy = 1\,dt$

$$\int \frac{dx}{\sin x} = \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2 \cdot \sin\frac{x}{2} \cdot \cos\frac{x}{2}}\,dx = \frac{1}{2}\left(\int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx + \int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx\right)$$

$= \frac{1}{2}\left(\int \operatorname{tg}\frac{x}{2}\,dx + \int \operatorname{ctg}\frac{x}{2}\,dx\right) = \frac{1}{2}\left(-2\ln\left(\cos\frac{x}{2}\right) + 2\ln\left(\sin\frac{x}{2}\right)\right)$

$= \ln\sin\frac{x}{2} - \ln\cos\frac{x}{2} = \ln\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \ln\left(\operatorname{tg}\frac{x}{2}\right) + \ln c$

$\Rightarrow \ln(\ln y) = \ln\left(\operatorname{tg}\frac{x}{2}\right) + \ln c$

$\ln y = c \cdot \operatorname{tg}\frac{x}{2}$

$\boxed{y(x) = e^{c\cdot \operatorname{tg}\frac{x}{2}}}$, $c \in \mathbb{R}$

general solution

Remark: In this case the singular sol. is included in the gen. sol.

$y\left(\frac{\pi}{2}\right) = 1 \Rightarrow e^{c\cdot \operatorname{tg}\frac{\pi}{4}} = 1 \Rightarrow c\cdot 1 = 0 \Rightarrow c = 0 \Rightarrow \boxed{y(x) = 1}$ is the IVP sol

c) $\begin{cases} x^2y' = y^2 - xy \\ y(-1) = -1 \end{cases} \Rightarrow \boxed{y' = \frac{y^2 - xy}{x^2} = \left(\frac{y}{x}\right)^2 - \frac{y}{x}}$ homogeneous diff. eq. in the Euler sense

subst: $z = \frac{y}{x} \Rightarrow y = z\cdot x$ $\quad y(x) = x\cdot z(x)$

$y' = z + z'\cdot x$

$\Rightarrow z + z'\cdot x = z^2 - z$

$z'\cdot x = z^2 - 2z$

$\boxed{z' = (z^2 - 2z)\cdot\frac{1}{x}}$ separable $\quad g(z) = 0 \Rightarrow z^2 - 2z = 0$

$z(z-2) = 0 \Rightarrow \begin{matrix} z(x) = 0 \\ z(x) = 2 \end{matrix}$ singular sol.

$g(z)$ $f(x)$

$\int \frac{dz}{z^2 - 2z} = \int \frac{dx}{x} \Rightarrow \ldots\ldots = \ln x + \frac{1}{2}\ln c$

$\int \frac{dz}{z(z-2)} = \int -\frac{1}{2}\cdot\frac{1}{z}\,dz + \int \frac{1}{2}\,\frac{1}{z-2}\,dz = -\frac{1}{2}\ln z + \frac{1}{2}\ln z-2$

$z-2$ z

$\frac{A}{z} + \frac{B}{z-2} \Rightarrow (z-2)A + zB = 1 \Rightarrow z(A+B) - 2A = 1$

$A + B = 0 \Rightarrow B = \frac{1}{2}$

$-2A = 1 \Rightarrow A = -\frac{1}{2}$

$-\frac{1}{2}\ln z + \frac{1}{2}\ln z - 2 = \ln x + \frac{1}{2}\ln c \quad /\cdot 2$

$-\ln e + \ln z - 2 = 2\ln x + \ln c$

$\ln \frac{z-2}{z} = \ln x^2 \cdot c$

$\frac{z-2}{z} = x^2 \cdot c \Rightarrow z-2 = x^2 \cdot c \cdot z \Rightarrow z(1 - c \cdot x^2) = 2$

$z(x) = \frac{2}{1 - c \cdot x^2}, \; c \in \mathbb{R}$

$y = x \cdot z \Rightarrow y(x) = \frac{2x}{1 - cx^2}, \; c \in \mathbb{R}$

$z(x) = 0 \Rightarrow y(x) = 0$ sing. sol

$z(x) = 2 \Rightarrow y(x) = 2x$ sing. sol

$y(-1) = -1 \Rightarrow y(x)$ not a sol. of IVP

$y(-1) = -1 \Rightarrow \frac{-2}{1-c} = -1 \Rightarrow -1 + c = -2$

$c = -1$

$\Rightarrow \boxed{y(x) = \frac{2x}{1+x^2}}$ sol. of IVP

d) $\begin{cases} xy' + y = e^x \\ y(a) = b \end{cases} \quad a, b \in \mathbb{R}, a \neq c$

$xy' + y = e^x$

$y' + \frac{y}{x} = \frac{e^x}{x}$

$y' + \frac{y}{x} = 0 \Rightarrow y' = -\frac{y}{x}$

$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x} \Rightarrow \ln y = -\ln x + c$

$\ln y = \ln \frac{1}{x} \cdot c$

$\boxed{y = \frac{c}{x}}$

$y_p = \frac{c(x)}{x}$

$y_p' = \frac{c'(x)\cdot x - c(x)}{x^2} = \frac{c'(x)}{x} - \frac{c(x)}{x^2}$

$c'(x) - \cancel{x}\frac{c(x)}{x^2} + \frac{c(x)}{x} = e^x$

$c'(x) = e^x \Rightarrow c(x) = e^x$

$y_p = \frac{e^x}{x}$

$y = y_p + y_o = \frac{e^x}{x} + \frac{c}{x} = \frac{e^x + c}{x}$, $c \in \mathbb{R}$

$y(a) = b \Rightarrow \frac{e^a + c}{a} = b \Rightarrow c = b\cdot a - e^a$

$y = \frac{e^x + b\cdot a - e^a}{x}$ general sol. of IVP

2) $\begin{cases} y' + \frac{y}{x} = \frac{1}{x^2 y^2} \quad (y^{-2}) \\ y(1) = 0 \end{cases}$

$z = y^{1-(-2)} \Rightarrow z = y^3 \Rightarrow y = z^{\frac{1}{3}}$

$y' = \frac{1}{3}\cdot z^{\frac{-2}{3}}\cdot z'$

$\frac{1}{3}\cdot z^{\frac{-2}{3}}\cdot z' + \frac{1}{x}\cdot z^{\frac{1}{3}} = \frac{1}{x^2}\, z^{\frac{-2}{3}} \quad / (1^{\frac{3}{\alpha}})\cdot z^{\frac{3}{3}} \Rightarrow$

$z' + \frac{3}{x}\cdot z = \frac{3}{x^2}$

$z' + \frac{3}{x} z = 0$ — $z = 0$ singular sol $\Rightarrow y = 0$ singular sol

$z' = -\frac{3}{x} z \Rightarrow \frac{dz}{dx} = -\frac{3}{x} z \Rightarrow \frac{dz}{z} = -\frac{3}{x} dx \Rightarrow \ln z = -3 \ln x + C \Rightarrow$

$\Rightarrow z_o = x^{-3}\cdot c$

$z_p = x^{-3}\cdot c(x)$

$z_p' = -3x^{-4}\cdot c(x) + c'(x)\, x^{-3}$

$\cancel{-3x^{-4}\cdot c(x)} + c'(x) x^{-3} + \cancel{\frac{3}{x}\cdot x^{-3}\cdot c(x)} = \frac{3}{x^2}$

$c'(x) x^{-3} = \frac{3}{x^2} \quad / \cdot x^3 \Rightarrow c'(x)\cdot\frac{1}{x} = 3 \quad / \cdot x \Rightarrow c'(x) = 3x$

$c(x) = \frac{3}{2}x^2$

$z_p = x^{-3}\cdot\frac{3}{2}x^2 = \frac{3}{2}\cdot\frac{1}{x}$

$z = z_0 + z_p = x^{-3} \cdot C + \frac{3}{2} \cdot \frac{1}{x} = x^{-3} \cdot C + \frac{3}{2} x^{-1}$

$z = y^3 \Rightarrow y^3 = x^{-3} \cdot C + \frac{3}{2} x^{-1}$

$y = \sqrt[3]{x^{-3} \cdot C + \frac{3}{2} x^{-1}}$

$y(1) = 0 \Rightarrow 0 = \sqrt[3]{C + \frac{3}{2}} \Rightarrow C = -\frac{3}{2}$

$\boxed{y = \sqrt[3]{x^{-3} \cdot -\frac{3}{2} + \frac{3}{2} x^{-1}}}$ general solution

2. Find the solutions for:

a) $\begin{cases} y'' = x e^{-x} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

b) $\begin{cases} y''' - \frac{2}{x} y'' = x^3 \\ y(1) = 1 \\ y'(1) = \frac{11}{10} \\ y''(1) = \frac{1}{2} \end{cases}$

c) $\begin{cases} x^2 y y'' - (y - xy')^2 = 0 \\ y(1) = \frac{1}{e} \\ y'(1) = \frac{2}{e} \end{cases}$

a) $\begin{cases} y'' = x e^{-x} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

$y'' = x e^{-x} \Rightarrow y' = \int x e^{-x} dx = -e^{-x} \cdot x + \int e^{x} dx = -e^{-x} \cdot x + e^{-x} + C_1$

$y = \int -e^{-x} \cdot x + e^{x} + C_1 \, dx = -e^{-x} \cdot x - e^{-x} - e^{-x} + x C_1 + C_2 =$

$= e^{-x} \cdot x - 2e^{-x} + x C_1 + C_2$

$y(0) = 0 \Rightarrow 0 - 2 + 0 + C_2 = 0 \Rightarrow C_2 = 2$

$y'(0) = 0 \Rightarrow 0 + 1 + C_1 = 0 \Rightarrow C_1 = -1$

$y = e^{-x} \cdot x - 2e^{-x} - x + 2$

b) $\begin{cases} y''' - \frac{2}{x} y'' = x^3 \\ y(1) = 1 \\ y'(1) = \frac{11}{10} \\ y''(1) = \frac{1}{2} \end{cases}$

$y''' - \frac{2}{x} y'' = x^3$

$y'' = z \Rightarrow y''' = z'$

$\Rightarrow \boxed{z' - \frac{2}{x} z = x^3}$ linear eq.

$z' - \frac{2}{x} z = 0$

$\frac{dz}{dx} = \frac{2}{x} z \Rightarrow \frac{dz}{z} = \frac{2dx}{x} \Rightarrow \ln z = 2 \ln x + c \Rightarrow$

$\ln z = \ln x^2 \cdot c \Rightarrow z_0 = x^2 \cdot c$

$z_p = x^2 \cdot c(x)$

$z_p' = 2x \cdot c(x) + c'(x) x^2$

$2x \cdot c(x) + c'(x) x^2 - 2x c(x) = x^3$

$c'(x) x^2 = x^3 \Rightarrow c'(x) = x \Rightarrow c(x) = \frac{x^2}{2}$

$z_p = x^2 \cdot \frac{x^2}{2} = \frac{x^4}{2}$

$z = z_p + z_0 = \frac{x^4}{2} + x^2 \cdot c \quad , c \in \mathbb{R}$

$y'' = \frac{x^4}{2} + x^2 \cdot c \Rightarrow y' = \int \frac{x^4}{2} + x^2 \cdot c \, dx = \frac{x^5}{10} + \frac{x^3}{3} c + c_1$

$y = \int \frac{x^5}{10} + \frac{x^3}{3} c + c_1 = \frac{x^6}{60} + \frac{x^4}{12} c + x c_1 + c_2$

$y''(1) = \frac{1}{2} \Rightarrow \frac{1}{2} + c = \frac{1}{2} \Rightarrow c = 0$

$y'(1) = \frac{11}{10} \Rightarrow \frac{1}{10} + c_1 = \frac{11}{10} \Rightarrow c_1 = 1$

$y(1) = 1 \Rightarrow \frac{1}{60} + 1 + c_3 = 1 \Rightarrow c_3 = -\frac{1}{60}$

$\boxed{y = \frac{x^6}{60} + x - \frac{1}{60}}$ IVP sol

c) $\begin{cases} x^2 y y'' - (y - xy')^2 = 0 \\ y(1) = \frac{1}{e} \\ y'(1) = \frac{2}{e} \end{cases}$

$x^2 y y'' - (y^2 - 2xyy' + x^2 y'^2) = 0$

$x^2 y y'' - y^2 + 2xyy' - x^2 y'^2 = 0 \quad / : y^2$

$x^2 \frac{y''}{y} - 1 + 2x\frac{y'}{y} - x^2\left(\frac{y'}{y}\right)^2 = 0$ $\qquad F(x, \frac{y'}{y}, \frac{y''}{y}) = 0$

$z = \frac{y'}{y}$

$z' = \frac{y''\cdot y - y'\cdot y'}{y^2} = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2 \Rightarrow \frac{y''}{y} = z' + \left(\frac{y'}{y}\right)^2 = z' + z^2$

$x^2(z' + z^2) - 1 + 2xz - x^2 z^2 = 0$

$x^2 z' + x^2 z^2 - 1 + 2xz - x^2 z^2 = 0$

$x^2 z' + 2xz = 1 \quad / : x^2$

$z' + \frac{2z}{x} = \frac{1}{x^2}$

$z' + \frac{2z}{x} = 0$

$z' = -\frac{2z}{x} \Rightarrow \frac{dz}{dx} = -\frac{2z}{x} \Rightarrow \frac{dz}{z} = -\frac{2\,dx}{x} \Rightarrow \ln z = \ln x^{-2}\cdot c$

$z_o = x^{-2}\cdot c$

$z_p = x^{-2}\cdot c(x)$

$z'_p = -2x^{-3}\cdot c(x) + c'(x)x^{-2}$

$-2x^{-3}c(x) + c'(x)x^{-2} + \frac{2x^{-2}\cdot c(x)}{x} = \frac{1}{x^2}$

$c'(x)x^{-2} = \frac{1}{x^2} \Rightarrow c'(x) = 1 \Rightarrow c(x) = x$

$z_p = x^{-1}$

$z = x^{-1} + x^{-2}\cdot c \Rightarrow \frac{y'}{y} = x^{-1} + x^{-2}c \qquad \frac{y'(1)}{y(1)} = 1 + c = \frac{\frac{2}{e}}{\frac{1}{e}} = 2 \Rightarrow c = 1$

$\frac{y'}{y} = x^{-1} + x^{-2}$

$y' = (x^{-1} + x^{-2})y \Rightarrow \frac{dy}{dx} = (x^{-1} + x^{-2})y \Rightarrow \frac{dy}{y} = dx(x^{-1} + x^{-2})$

$$\ln y = \ln x + \frac{x^{-1}}{-1} + c$$

$$\ln y = \ln x + \ln e^{-\frac{1}{x}} + \ln c$$

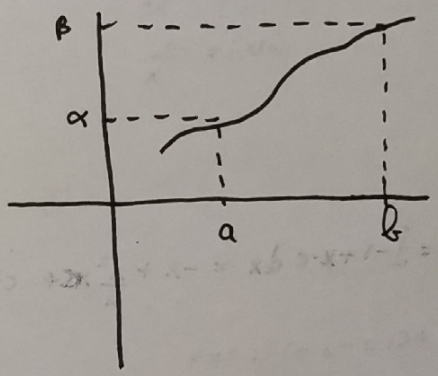
$$y = x \cdot e^{-\frac{1}{x}} \cdot c$$

$$y(1) = \frac{1}{e} = 1 \cdot e^{-1} \cdot c \Rightarrow c = 1$$

$$\boxed{y(x) = x \cdot e^{-\frac{1}{x}}}$$ general solution

Boundary value problem for second diff. eq.

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha \\ y(b) = \beta \end{cases}$$



3) Find

a) $\begin{cases} y'' = 2 \\ y(0) = 1 \\ y(1) = 3 \end{cases}$

b) $\begin{cases} xy'' - y' = 1 \\ y(0) = -4 \\ y(2) = 2 \end{cases}$

c) $\begin{cases} y'' = y' \ln y' \\ y(0) = 0 \\ y(1) = 1 \end{cases}$

a) $y'' = 2$

$y' = 2x + c$

$y = \frac{2x^2}{2} + cx + c_1 = x^2 + cx + c_1$

$y(0) = 1 \Rightarrow 0 + 0 + c_1 = 1 \Rightarrow c_1 = 1$

$y(1) = 3 \Rightarrow 1 + c + 1 = 3 \Rightarrow c = 1$

$y = x^2 + x + 1$

b) $\begin{cases} xy'' - y' = 1 \\ y(0) = -4 \\ y(2) = 2 \end{cases}$

$y' = z \Rightarrow \quad x z' - z = 1$

$z' - \frac{z}{x} = \frac{1}{x}$

$z' - \frac{z}{x} = 0 \Rightarrow \frac{dz}{dx} = \frac{z}{x} \Rightarrow \frac{dz}{z} = \frac{dx}{x} \Rightarrow z = x \cdot c$

$z_p = x \cdot c(x)$

$z_p' = c(x) + c'(x) \cdot x$

$c(x) + c'(x) \cdot x - c(x) = \frac{1}{x} \Rightarrow c'(x) = \frac{1}{x^2}$

$c(x) = -\frac{1}{x}$

$z_p = x \cdot -\frac{1}{x} = -1$

$z = -1 + x \cdot c$

$y' = -1 + x \cdot c \Rightarrow y = \int -1 + x \cdot c \, dx = -x + \frac{x^2}{2} c + c_1 =$

$y(0) = -4 \Rightarrow 0 + 0 + c_1 = -4 \Rightarrow c_1 = -4$

$y(2) = 2 \Rightarrow -2 + 2c = 2 \Rightarrow c = 2$

$\boxed{y = -x + x^2 - 4}$

c) $\begin{cases} y'' = y' \ln y' \\ y(0) = 0 \\ y(1) = 1 \end{cases}$

$z = y' \Rightarrow z' = y''$

$z' = z \ln z \Rightarrow \frac{dz}{dx} = z \ln z \Rightarrow \frac{dz}{z \ln z} = dx \Rightarrow \int \frac{1}{z} \cdot \frac{1}{\ln z} dz = x + c \Rightarrow$

$t = \ln z$

$\Rightarrow \int \frac{1}{t} dt = x + c \Rightarrow \ln \ln z = x + c$

$\ln z = e^x \cdot c$

$z = e^{ex \cdot c}$

$y' = e^{ex \cdot c} \Rightarrow y = \int e^{ex \cdot c} dx = \frac{e^{ex \cdot c}}{e^x \cdot c} + c_1$

$y(0) = 0 \Rightarrow \frac{e^c}{c} + c_1 = 0 \Rightarrow c_1 = -\frac{e^c}{c}$

$y(2) = 2 \Rightarrow \frac{e^{e^2 \cdot c}}{e^2 \cdot c} - \frac{e^c}{c} = 2$???