SEMINAR 7
ECUAȚII

THE CAUCHY PROBLEM. THE EXISTENCE AND UNIQUENESS THEOREMS

(1) $\begin{cases} y' = f(x,y) \\ (2)\ y(x_0) = y^0 \end{cases}$ $\quad f:[a,b]\times\mathbb{R}\to\mathbb{R}$, $x_0\in[a,b]$, $y^0\in\mathbb{R}$

(1) + (2) $\Leftrightarrow$ (3)

(3) $y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$ the Volterra integral

THEOREM 1 (The ∃! theorem in the space)

Let us consider the problem (1) + (2). We suppose that:

(i) $f\in C([a,b]\times\mathbb{R},\mathbb{R})$

(ii) f is **LIPSCHITZ** with respect to the second variable, i.e.

$\exists L_f>0$ such that $|f(x,u)-f(x,v)|\le L_f\cdot|u-v|$, $\forall u,v\in\mathbb{R}$

Then the problem (1) + (2) has an unique solution $y^*\in C([a,b],\mathbb{R})$

REMARK. a) Let there be a function $g\in C'(\mathbb{R})$
If there exists $M>0$ s.t. $|g'(x)|\le M$, $\forall x\in\mathbb{R}$
then g is lipschitz with $L_g = M$

b) Let $f\in C'([a,b]\times\mathbb{R},\mathbb{R})$. If there exists $M>0$ such that
$\left|\frac{\partial f}{\partial y}(x,y)\right|\le M$, $\forall (x,y)\in[a,b]\times\mathbb{R}$ Then
f is lipschitz w.r.t. the second variable y, with $L_f = M$

THEOREM 2 (The ∃! theorem in the ball)

Let's consider the problem (1)+(2)

$f: D_f \to \mathbb{R}$, $D_f \subseteq \mathbb{R}^2$ domain (convex and open set)

$\bar{D} = [x_0 - a, x_0 + a] \times [y^0 - b, y^0 + b]$, $a, b > 0$

Suppose that:

(i) $f \in C(\bar{D}, \mathbb{R})$

(ii) f is lipschitz w.r.t. the second variable on $\bar{D}$.

$\iff \exists L_f > 0$ s.t. $|f(x,u) - f(x,v)| \leq L_f \cdot |u - v|$, $\forall x \in [x_0 - a, x_0 + a]$

$\forall u, v \in [y^0 - b, y^0 + b]$

$\Updownarrow$

$(x,u), (x,v) \in \bar{D}$

Then the problem (1)+(2) has unique solution $y^* \in C([x_0 - h, x_0 + h], [y^0 - b, y^0 + b])$ where

$h = \min\{a, \frac{b}{M_f}\}$, $M_f = \max_{\bar{D}} |f(x,y)|$

Exercise 1 Give an existence and uniqueness result for the following Cauchy problems

a) $\begin{cases} y' = \sin(xy) + 2y \\ y(0) = 0 \end{cases}$

b) $\begin{cases} xy' = y - 1 \\ y(1) = 0 \end{cases}$

c) $\begin{cases} y' = 3x + 4y^3 \\ y(0) = 0 \end{cases}$

d) $\begin{cases} y' = e^{-x} + y^2 \\ y(0) = 1 \end{cases}$

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a) $\begin{cases} y' = \sin(xy) + 2y \\ y(0) = 0 \end{cases}$

$x_0 = 0 \quad y^0 = 0$

$f(x,y) = \sin(xy) + 2y \qquad f: \mathbb{R}^2 \to \mathbb{R}$

$I = [-a, a]$, $a > 0$

f is cont on $[-a, a] \times \mathbb{R}$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|\cos(xy)\cdot x + 2\right| \leq \left|\cos(xy)\cdot x\right| + 2 =$$

$$= \underbrace{|x|}\cdot \underbrace{|\cos(xy)|}_{\leq 1} + 2 \leq \underline{a+2}$$

$x \in [-a, a] \iff |x| \leq a$

$\Rightarrow \frac{\partial f}{\partial y}$ is bounded on $[-a, a] \times \mathbb{R} \Rightarrow f$ is lipschitz w.r.t. variable y on $[-a, a] \times \mathbb{R}$ $(L_f = a + 2) \overset{Th_1}{\Rightarrow} \exists!\, y^* \in C([-a, a], \mathbb{R})$

Theorem $\forall a > 0$ the Cauchy problem (a) has a unique solution $y^* \in C([-a, a], \mathbb{R})$

b) $\begin{cases} xy' = y - 1 \\ y(1) = 0 \end{cases} \iff \begin{cases} y' = \frac{y-1}{x} \\ y(1) = 0 \end{cases}$

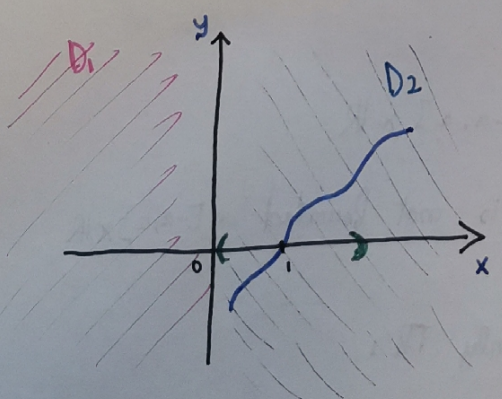
$x_0 = 1 \quad y^0 = 0$

$f(x, y) = \frac{y-1}{x}$ $\quad f : \mathbb{R}^* \times \mathbb{R} \longrightarrow \mathbb{R}$

$f : ((-\infty, 0) \cup (0, \infty)) \times \mathbb{R}) \longrightarrow \mathbb{R}$

$f : \underbrace{((-\infty, 0) \times \mathbb{R})}_{D_1} \cup \underbrace{((0, \infty) \times \mathbb{R})}_{D_2} \longrightarrow \mathbb{R}$

$(x_0, y^0) \in D_2$



We take $I = [a, b]$ s.t. $x_0 = 1 \in [a, b]$

$\iff 0 < a \leq 1$ and $b > 1$

$f : [a, b] \times \mathbb{R} \to \mathbb{R}$ $\quad f \in C([a, b] \times \mathbb{R}, \mathbb{R})$

$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|\frac{1}{x}\right|$

$x \in [a,b]$
$0 < a \le x \le b$ $\Rightarrow \left|\frac{\partial f}{\partial y}(x,y)\right| = \frac{1}{x} \le \frac{1}{a}$

$\Rightarrow \frac{1}{b} \le x \le \frac{1}{a}$

$\Rightarrow \frac{\partial f}{\partial y}$ is bounded on $[a,b] \times \mathbb{R}$

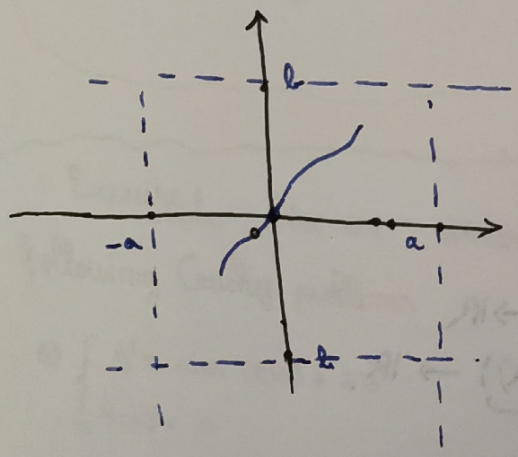
$\Rightarrow f$ is lipschitz w.r.t y on $[a,b] \times \mathbb{R}$ $(Lf = \frac{1}{a})$

$\Rightarrow \exists!\ y^* \in C([a,b], \mathbb{R})$

Theorem: If $0 < a \le 1$ and $b > 1$ then the Cauchy problem (b) has a unique solution $y^* \in C([a,b], \mathbb{R})$

c) $\begin{cases} y' = 3x + 4y^3 \\ y(0) = 0 \end{cases}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$x_0 = 0 \quad y_0 = 0$



$f(x,y) = 3x + 4y^3 \Rightarrow f$ continuous on $\mathbb{R}^2$

$I \subseteq \mathbb{R}$ s.t. $x_0 = 0 \in I$

$a > 0 \Rightarrow 0 \in [-a,a] = I$ $\quad f$ is cont on $[-a,a] \times \mathbb{R}$

$\left|\frac{\partial f}{\partial y}(x,y)\right| = |12y^2| = 12y^2 \longrightarrow \infty$ as $|y| \longrightarrow \infty$ $\quad \frac{\partial f}{\partial y}$ is not bounded on $[-a,a] \times \mathbb{R}$

$(x,y) \in [-a,a] \times \mathbb{R}$; $y \in \mathbb{R}$ we cannot apply TH 1

We consider $\bar{D} = [-a, a] \times [-b, b]$, $a, b > 0$

$f \in C(\bar{D}, \mathbb{R})$

f Lipschitz w.r.t. y on $\bar{D}$?

$$\left| \frac{\partial f}{\partial y}(x, y) \right| = 12 y^2 \leq 12 b^2$$

$(x, y) \in \bar{D} = [-a, a] \times [-b, b] \Rightarrow y \in [-b, b]$

$$-b \leq y \leq b \iff |y| \leq b$$
$$y^2 \leq b^2$$

$\Rightarrow \frac{\partial f}{\partial y}$ is bounded on $\bar{D} \Rightarrow$

$\Rightarrow f$ is Lipschitz w.r.t. y on $\bar{D}$

$(L_f = 12 \cdot b^2)$

$\xRightarrow[Th_2]{} \exists!\ y^* \in C([-h, h], [-b, b])$

where $h = \min\left\{a, \frac{b}{M_f}\right\}$

$M_f = \max\limits_{\bar{D}} |f(x, y)|$

$|f(x, y)| = |3x + 4y^3| \leq 3|x| + 4|y|^3 \leq 3 \cdot a + 4 b^3$

$x \in [-a, a]$ $\quad y \in [-b, b]$

$\Rightarrow h = \min\left\{a, \frac{b}{3a + 4b^3}\right\}$

Theorem If $a > 0$ $b > 0$ then the IVP (C) has a unique sol y^* $y^* \in C([-h, h], [-b, b])$ where $h = \min\left\{a, \frac{b}{3a + 4b^3}\right\}$

For example if we take $a = 1$ $b = 1$ then:

$h = \min\left\{1, \frac{1}{7}\right\} = \frac{1}{7} \Rightarrow \exists!\ y^* \in C\left([-\frac{1}{7}, \frac{1}{7}], [-1, 1]\right)$