

LINEAR DIFFERENTIAL EQUATIONS

$$(1) \underbrace{y^{(m)} + a_1(x)y^{(m-1)} + \dots + a_{m-1}(x)y' + a_m(x)y}_{\text{Linear part}} = f(x) \quad f, a_1, \dots, a_m \in C(I) \quad \text{not continuous}$$

$f \neq 0 \Rightarrow$ nonhomogeneous linear diff. eq.

$f = 0 \Rightarrow$ homogeneous linear diff. eq.

$$(2) y^{(m)} + a_1(x)y^{(m-1)} + \dots + a_{m-1}(x)y' + a_m(x)y = 0$$

$$L: C^m(I) \rightarrow C(I)$$

derivatela pînă la ordinul m și funcția rezultă continuă
 $y \mapsto Ly = y^{(m)} + a_1 y^{(m-1)} + \dots + a_m y$

L is linear operator

$$(1) \Leftrightarrow Ly = f(x)$$

$$S = \text{Ker } L + \{y_p\}$$

$$\text{Ker } L = \{y \mid Ly = 0\} \text{ set of (2)}$$

y_p is a particular sol of (1)

THEOREM $\dim \text{Ker } L = m$, $\text{Ker } L$ is a linear subspace of $C(I)$

$$\exists \{y_1, \dots, y_m\} \subset \text{Ker } L \text{ a basis in } \text{Ker } L \Leftrightarrow$$

$$\Leftrightarrow \{y_1, \dots, y_m\} \subset \text{Ker } L \text{ and } \{y_1, \dots, y_m\} \text{ linear independent vps}$$

$$\Leftrightarrow \{y_1, \dots, y_m\} \subset \text{Ker } L \text{ and } W(x, y_1, \dots, y_m) \neq 0$$

$$W(x, y_1, \dots, y_m) = \begin{vmatrix} y_1 & \dots & y_m \\ y_1' & \dots & y_m' \\ \vdots & & \vdots \\ y_1^{(m-1)} & \dots & y_m^{(m-1)} \end{vmatrix} \quad \text{the WRONSKIAN}$$

THEOREM

a) If $y_1, \dots, y_m \in C^m(I)$ are linearly dependent $\Rightarrow$
 $\Rightarrow W(x; y_1, \dots, y_m) = 0$

b) If $y_1, \dots, y_m \in \text{Ker } L$ are linearly independent $\Rightarrow$
 $\Rightarrow W(x; y_1, \dots, y_m) \neq 0 \quad \forall x \in I$

Ex1 $W(x; y_1, \dots, y_m) = 0$ on $I = [a, b] \Rightarrow$
Are the functions $y_1, \dots, y_m$ linearly dependent?

Answer is NO

$$y_1(x) = \begin{cases} x^2 & x \in [-1, 0) \\ 0 & x \in [0, 1] \end{cases}$$

$$I = [-1, 1]$$

$$m = 2$$

$$y_2(x) = \begin{cases} 0 & x \in [-1, 0) \\ x^2 & x \in [0, 1] \end{cases}$$

$$y_1, y_2 \in C^1(I)$$

$$W(x; y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{cases} \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix} & x \in [-1, 0) \\ \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix} & x \in [0, 1] \end{cases} = 0$$

$$\Rightarrow W(x; y_1, y_2) = 0 \text{ on } [-1, 1]$$

$\{y_1, y_2\}$ is a linear independent system of functions

$$c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 = c_2 = 0$$

$$c_1 y_1 + c_2 y_2 = 0$$

$$c_1 \cdot y_1(x) + c_2 \cdot y_2(x) = 0 \quad \forall x \in [-1, 1]$$

$$\text{if } x \in [-1, 0) \text{ then } c_1 x^2 + c_2 \cdot 0 = 0 \Rightarrow c_1 x^2 = 0 \quad \forall x \in [-1, 0)$$

$$\Rightarrow c_1 = 0$$

$$\text{if } x \in [0, 1] \text{ then } c_2 \cdot y_2(x) = 0$$

$$c_2 x^2 = 0 \Rightarrow c_2 = 0$$

$\Rightarrow y_1, y_2$ are linearly independent

Ex2. Find the linear homogeneous diff eq. which has as a fundam. syst. of solutions the functions:

$$y_1(x) = e^x \cos 3x$$

$$y_2(x) = e^x \sin 3x$$

$y \in \text{KerL}$ if $\{y_1, y_2\}$ is a fundam. syst. of sol $\Rightarrow$

$\dim \text{KerL} = 2 \Rightarrow$ the diff eq has order 2

$$y'' + a_1(x) \cdot y' + a_2(x)y = 0$$

if $y \in \text{KerL} \Rightarrow \{y, y_1, y_2\}$ is a linear dependent system

$$\Rightarrow W(x; y, y_1, y_2) = 0$$

$$\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0 \quad \forall x \in I$$

$$\begin{vmatrix} y & e^x \cos 3x & e^x \sin 3x \\ y' & e^x \cos 3x - 3e^x \sin 3x & e^x \sin 3x + 3e^x \cos 3x \\ y'' & e^x \cos 3x - 6e^x \sin 3x + 9e^x \cos 3x & e^x \sin 3x + 6e^x \cos 3x - 9e^x \sin 3x \end{vmatrix} = 0$$

$$\Rightarrow \dots \Rightarrow y'' - 2y' + 10y = 0$$

Ex3 Consider the diff eq. $Ly = 0$ where $Ly = y'' + a_1 y' + a_2 y$, $a_1, a_2 \in C(I)$

Make the change of variables $y = y_1 z$ where y_1 is given, $y_1 \in C^2(I)$.

Conclusion!

$$(y'') + a_1 y' + a_2 y = 0$$

$y = y(x)$ is the unknown function

$$y(x) \mapsto z(x)$$

$$y(x) = y_1(x) z(x) \quad \text{where } y_1 \in C^2(I) \text{ is given}$$

$$y' = y_1' z + y_1 z' \quad | \cdot a_1$$

$$y'' = y_1'' z + y_1' z' + y_1' z' + y_1 z'' = y_1'' z + 2y_1' z' + y_1 z''$$

$$\Rightarrow y_1'' z + 2y_1' z' + y_1 z'' +$$

$$a_1 y_1' z + a_1 y_1 z' +$$

$$a_2 y_1 z = 0 \Rightarrow$$

$$\Rightarrow y_1 z'' + z'(2y_1' + a_1 y_1) + z(y_1'' + a_1 y_1' + a_2 y_1) = 0$$

$$Ly_1 = 0$$

if y_1 is a solution of the eq

$y'' + a_1 y' + a_2 y = 0$ then $Ly_1 = 0$ and

$y_1 z'' + (2y_1' + a_1 y_1) z' = 0$ second order diff eq. of the form $F(z'', z') = 0$

$$z' = u$$

$$\Rightarrow y_1 u' + (2y_1' + a_1 y_1) u = 0 \quad \text{first order homogeneous linear eq}$$

Conclusion: The minimal requirement to solve a homog. linear second order diff. eq is to find one solution.

Ex4 Solve the following diff eq:

a) $x y'' - (2x+1)y' + 2y = 0$ knowing that admits sol of the form $y_1(x) = e^{rx}$

b) $x y'' - (x+3)y' + 2y = 0$ knowing that it admits a solution which is a polynomial of degree 2

Homework

$$a) xy'' - (2x+1)y' + 2y = 0$$

$$y_1(x) = e^{\alpha x} \quad \alpha = ?$$

$$y_1'(x) = \alpha e^{\alpha x}$$

$$y_1''(x) = \alpha^2 e^{\alpha x}$$

$$xy_1'' - (2x+1)y_1' + 2y_1 = 0 \Rightarrow x\alpha^2 e^{\alpha x} - (2x+1)\alpha e^{\alpha x} + 2e^{\alpha x} = 0$$

$$x\alpha^2 - (2x+1)\alpha + 2 = 0 \Rightarrow x\alpha^2 - 2x\alpha - \alpha + 2 = 0 \Rightarrow$$

$$\Rightarrow x\alpha(\alpha-2) - (\alpha-2) = 0 \Rightarrow (\alpha-2)(x\alpha-1) = 0 \quad \forall x$$

$$\alpha-2=0 \Rightarrow \alpha=2 \Rightarrow y_1(x) = e^{2x}$$

când avem tipul ăsta de ecuație cu soluție dată facem substituția:

$$\boxed{y = y_1 \cdot z} \Rightarrow y = e^{2x} \cdot z$$

$$y' = 2e^{2x} \cdot z + e^{2x} \cdot z'$$

$$y'' = 4e^{2x} \cdot z + 2e^{2x} \cdot z' + 2e^{2x} \cdot z' + e^{2x} \cdot z'' = 4e^{2x} \cdot z + 4e^{2x} \cdot z' + e^{2x} \cdot z''$$

$$\boxed{xy'' - (2x+1)y' + 2y = 0}$$

$$4xe^{2x} \cdot z + 4xe^{2x} \cdot z' + xe^{2x} \cdot z'' - 2(2x+1)e^{2x} \cdot z - (2x+1)e^{2x} \cdot z' + 2e^{2x} \cdot z = 0 \Rightarrow$$

$$\Rightarrow xz'' + z' \cdot (4x - 2x - 1) + z \cdot (-4x - 2 + 4x + 2) = 0 \Rightarrow$$

$$\Rightarrow \boxed{xz'' + (2x-1)z' = 0}$$

$$z' = u \Rightarrow xu' + (2x-1)u = 0$$

$$u' = -\frac{(2x-1)}{x} \cdot u$$

$$u' = \frac{du}{dx}$$

$$\frac{du}{dx} = -\frac{(2x-1)}{x} \cdot u \Rightarrow$$

$$\frac{du}{u} = \left(-2 + \frac{1}{x}\right) dx \Rightarrow \int \frac{du}{u} = \int \left(-2 + \frac{1}{x}\right) dx \Rightarrow$$

$$\ln u = -2x + \ln x + \ln c$$

$$u(x) = c \cdot e^{-2x + \ln x}$$

$$\Rightarrow u(x) = c \cdot x \cdot e^{-2x}$$

$$z' = u \Rightarrow z' = c_1 x \cdot e^{-2x}$$

$$z(x) = \int c_1 x e^{-2x} dx + c_2$$

$$\Rightarrow z(x) = c_1 \left(-\frac{x}{2} e^{-2x} + \frac{1}{2} \int e^{-2x} dx \right) + c_2$$

$$\boxed{z(x) = c_1 \left(-\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \right) + c_2}$$

$$\Rightarrow y = e^{2x} \cdot z \Rightarrow y(x) = c_1 \left(-\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \right) \cdot e^{2x} + c_2 e^{2x}$$

$$y(x) = c_1 \left(-\frac{x}{2} - \frac{1}{4} \right) + c_2 e^{2x}, \quad c_1, c_2 \in \mathbb{R}$$

Ex 5. Solve the diff eq.:

$$(2x+1)y'' + 4x \cdot y' - 4y = (2x+1)^2$$

Knowing that the homogeneous eq. has the functions

$$y_1(x) = x \quad y_2(x) = e^{-2x} \text{ as solutions}$$

the general sol: $y = y_0 + y_p$

y_0 is the general solution of the homog.

$$\boxed{(2x+1)y'' + 4x \cdot y' - 4y = 0}$$

y_p is a particular solution of the nonhomogen. equation

y_1, y_2 are solutions of the homog. eq.

1) we check if $\{y_1, y_2\}$ is linear indep. syst. of functions

$$\Leftrightarrow W(x; y_1, y_2) \neq 0$$

$$W(x; y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & e^{-2x} \\ 1 & -2e^{-2x} \end{vmatrix} = x \cdot (-2e^{-2x}) - e^{-2x} = -2xe^{-2x} - e^{-2x} = -e^{-2x}(2x+1)$$

$$> 0 \quad \neq 0 \text{ if } x \neq -\frac{1}{2}$$

Remark: We solve our eq. on interval which doesn't contain $x = -\frac{1}{2}$

$\Rightarrow W(x, y_1, y_2) \neq 0 \Rightarrow \{y_1, y_2\}$ is a fundamental syst. of sol

$$\Rightarrow \boxed{y_0 = c_1 y_1 + c_2 y_2}$$

$$y_0(x) = c_1 x + c_2 e^{-2x}, \quad c_1, c_2 \in \mathbb{R}$$

$$y_p(x) = ?$$

$$y_p(x) = c_1(x) \cdot x + c_2(x) e^{-2x}$$

$$(2x+1)y_p'' + 4x \cdot y_p' - 4y_p = (2x+1)^2$$

$$y_p'(x) = \underline{c_1'(x) \cdot x} + c_1(x) + \underline{c_2' \cdot e^{-2x}} - 2c_2(x) e^{-2x}$$

We impose the cond that: $\boxed{c_1'x + c_2' e^{-2x} = 0}$

General condition

$$y_p^{(k+1)}(x) = c_1 \cdot y_1^{(k+1)} + \dots + c_m \cdot y_m^{(k+1)} = 0$$

$$\Rightarrow y_p'(x) = c_1(x) - 2c_2(x) \cdot e^{-2x}$$

$$y_p''(x) = c_1'(x) - 2c_2'(x) \cdot e^{-2x} + 4c_2(x) \cdot e^{-2x}$$

$$(2x+1) \cdot y_p'' + 4x \cdot y_p' - 4y_p = (2x+1)^2$$

$$(2x+1)c_1' - 2(2x+1)c_2' e^{-2x} + 4(2x+1)c_2 e^{-2x} + \underline{4c_1'x - 8c_2'x e^{-2x}} - \underline{4c_1'x - 4c_2'x e^{-2x}} = (2x+1)^2$$

$$(2x+1)c_1' - 2(2x+1)c_2' e^{-2x} + c_2(2x+4 - 2x - 4)e^{-2x} = (2x+1) \Rightarrow$$

$$\Rightarrow (2x+1)c_1' - 2(2x+1)c_2' e^{-2x} = (2x+1)^2 \Rightarrow$$

$$\Rightarrow c_1' - 2c_2' e^{-2x} = 2x+1$$

$$\begin{cases} c_1' - 2c_2' e^{-2x} = 2x+1 \\ c_1'x + c_2' e^{-2x} = 0 \quad / \cdot 2 \end{cases}$$

$$\begin{cases} c_1' - 2c_2' e^{-2x} = 2x+1 \\ 2c_1'x + 2c_2' e^{-2x} = 0 \end{cases}$$

$$c_1' + 2c_1'x = 2x+1$$

$$c_1'(2x+1) = 2x+1$$

$$c_1' = 1 \Rightarrow c_1 = x$$

$$x - 2c_2' e^{-2x} = 2x+1$$

$$x c_2' e^{-2x} = -x \Rightarrow c_2' = \frac{-1}{e^{-2x}} = -e^{2x} \Rightarrow c_2 = -\frac{e^{2x}}{2}$$

$$y = y_0 + y_p = c_1 x + c_2 e^{-2x} + x^2 + \frac{1}{2}$$

Ex 4/b

$$xy'' - (x+3)y' + 2y = 0$$

Solution is a polynomial of degree 2 $\Rightarrow y = ax^2 + bx + c$

$$y' = 2ax + b$$

$$y'' = 2a$$

$$x(2a) - (x+3)(2ax+b) + 2(ax^2+bx+c) = 0$$

$$2ax - (2ax^2 + xb + 6ax + 3b) + 2ax^2 + 2bx + 2c = 0$$

$$\underline{2ax} - \underline{2ax^2} - \underline{xb} - \underline{6ax} - \underline{3b} + \underline{2ax^2} + \underline{2bx} + \underline{2c} = 0$$

$$-4ax + bx - 3b + 2c = 0$$

$$(-4a + b)x - 3b + 2c = 0$$

$$\begin{cases} -4a + b = 0 \Rightarrow b = 4a \\ -3b + 2c = 0 \Rightarrow c = 6a \end{cases}$$

$$y_1 = ax^2 + 4ax + 6a$$

$$ax^2 + 4ax + 6a$$

$$y = y_1 \cdot z \Rightarrow y = (x^2 + 4x + 6) \cdot z \cdot a$$

$$y' = [(2x+4) \cdot z + z'(x^2+4x+6)] \cdot a$$

$$y'' = [2 \cdot z + (2x+4)z' + z''(x^2+4x+6) + (2x+4)z'] \cdot a$$

$$= [2z + (4x+8)z' + z''(x^2+4x+6)] \cdot a$$

$$x[2z + 2(2x+4)z' + z''(x^2+4x+6)] - (x+3)[(2x+4)z + z'(x^2+4x+6)] + 2(x^2+4x+6)z = 0$$

$$z[2x - (x+3)(2x+4) + 2(x^2+4x+6)] = z(2x - 2x^2 - 4x - 6x - 12 + 2x^2 + 8x + 12) = 0$$

$$z'[4x^2 + 8x - (x+3)(x^2+4x+6)] = z'(4x^2 + 8x - x^3 - 4x^2 - 6x - 3x^2 - 12x - 18) =$$

$$= z'(-x^3 - 10x - 3x^2 - 18)$$

$$z''(x^2+4x+6) + z'(-x^3 - 10x - 3x^2 - 18) = 0 \quad | : (x^2+4x+6)$$

$$z'' + z' \frac{(-x^3 - 10x - 3x^2 - 18)}{x^2+4x+6} = 0$$

$$z' = u \Rightarrow u' + u \frac{(-x^3 - 10x - 3x^2 - 18)}{x^2+4x+6} \Rightarrow u' = u \cdot \frac{x^3 + 10x + 3x^2 + 18}{x^2+4x+6}$$

$$u' = u \cdot \frac{x^3 + 10x + 3x^2 + 18}{x^2 + 4x + 6}$$

$$\frac{du}{dx} = u \cdot \frac{x^3 + 10x + 3x^2 + 18}{x^2 + 4x + 6} \Rightarrow \frac{du}{u} = \frac{x^3 + 10x + 3x^2 + 18}{x^2 + 4x + 6} dx$$

$\int = 1$

$$1 = \int \frac{x^3 + 6x^2 + 18x + 28}{3x^2 + 12x + 18} dx + \int \frac{-3x^2 - 12x - 18}{3x^2 + 12x + 18} dx + \int \frac{4x + 8}{x^2 + 4x + 6} dx =$$

$$t = x^3 + 6x^2 + 18x + 28$$

$$t_2 = x^2 + 4x + 6$$

$$-1$$

$$= \int t dt - x + \int \frac{2}{t_2} dt_2 = \frac{t^2}{2} - x + 2 \ln t_2 = \frac{(x^3 + 6x^2 + 18x + 28)^2}{2} - x + 2 \ln(x^2 + 4x + 6) + C$$

$$= \ln e^{\frac{(x^3 + 6x^2 + 18x + 28)^2}{2}} - \ln e^x + \ln (x^2 + 4x + 6)^2 =$$

$$= \ln e^{\frac{(x^3 + 6x^2 + 18x + 28)^2}{2x}} \cdot (x^2 + 4x + 6)^2$$

$$\ln u = \ln \left(e^{\frac{(x^3 + 6x^2 + 18x + 28)^2}{2x}} \cdot (x^2 + 4x + 6)^2 \right) \cdot C$$

$$u = e^{\frac{(x^3 + 6x^2 + 18x + 28)^2}{2x}} \cdot (x^2 + 4x + 6)^2 \cdot C$$

$$z' = e^{\frac{(x^3 + 6x^2 + 18x + 28)^2}{2x}} \cdot (x^2 + 4x + 6)^2 \cdot C$$

$$z = \int e^{\frac{(x^3 + 6x^2 + 18x + 28)^2}{2x}} \cdot (x^2 + 4x + 6)^2 \cdot C$$